Journal of Computing and Applied Informatics (JoCAI) Vol. 03, No. 1, 2019 | 18 - 34

DATA SCIENCE

Journal of Computing and Applied Informatics

On Mathematical Braids

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Abstract. A braid is any sequence of crossings of the n-strand braid with a positive number n, provided none of the strands are self-crossing. The idea is that braids can be organized into groups, in which the group operation is *, which means: “do the first braid on a set of strands, and then follow it with a second on the twisted strand”. This study dealt with the formulation of additional basic properties of mathematical braids and the connection of mathematical braids to exponential theorems. Moreover, the researchers developed a program that could generate the total number of crossings and the total number of generators in an n-strand braid with a positive number n.

Keyword: Braid, exponential theorems, generators, crossings.

Abstrak. *Braid adalah suatu urutan penyilangan jalinan n-untai dengan angka positif n, dengan aturan tidak ada untaian yang melintas sendiri. Idenya adalah bahwa braid dapat diatur ke dalam kelompok, di mana operasi kelompok adalah *, yang berarti: “masukan braid pada himpunan untaian, dan kemudian ikuti dengan braid yang kedua pada untaian yang terpuntir". Penelitian ini berhubungan dengan formulasi sifat-sifat dasar braids pada matematika dan hubungan braid pada matematika dengan teorema eksponensial. Selain itu, para peneliti mengembangkan program yang dapat menghasilkan jumlah total penyilangan dan jumlah total generator dalam jalinan n-untaian dengan angka positif n.*

***Kata kunci**: Braid, teorema eksponensial, generator, penyilangan.*

Received 22 December 2018 | Revised 24 January 2019 | Accepted 29 January 2019

1. Introduction

Mathematics originated from the Greek word "máthēma" which means knowledge, study, and learning. Mathematics is a broad-ranging field of study in which the properties and interactions of idealized objects are examined. It is also a branch of science that deals with the study of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, space, and change [21].

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ISSN: 2580-6769 | e-ISSN: 2580-829X | DOI: /10.32734/jocai.v3.i1-624

Journal Homepage: https://talenta.usu.ac.id/JoCAI

Topology is a branch of mathematics which deals with the shapes of geometric objects, which in applications may be as small as knotted DNA or long-chain polymers, or as large as the universe itself. It is also the mathematical study of the properties that are preserved through deformations, twisting, and stretching of objects without tearing [4][21].

The area of mathematics most suited to the investigation of braids is called topology. In topology, braid theory is an abstract geometric theory studying the everyday braid concept and some generalizations. The idea is that braids can be organized into groups, in which the group operation is “do the first braid on a set of strands, and then follow it with a second on the twisted strands”. Such braid groups may be described by explicit presentations, as was shown by E. Artin in 1925 [3][4].

Braid theory has been studied since the early 1920. It is founded by Emil Artin, a German mathematician in his original paper “Theorie der Zopfe” in 1925. Though his studies were initially motivated by the geometric constructions of braids, it was not long before the powerful algebra behind braid theory became evident. Since then, the theory has branched out into many fields of application, from encryption to solving polynomial equations. However, the study of braids in themselves is mathematically both rich and deep [7].

Braids can be thought of as a number of pieces of strand, which isvisualized by colored drawing lines thatmay cross over or cross under each other strand. The researchers used colored lines to easily identify which strand cross over or cross under.

Three properties or axioms must be satisfied to constructbraids. These three properties or axioms are used in order for the braids to be transcribed as a word in the paper “Exploration in Braid Theory” by [1]. The researchers used these properties or axioms in order for them to be more specific on their topic and also for the readers to clearly understand it. These three properties or axioms are: (1) no strands can be tangent or no strands intersect each other at any point, (2) only two strands can cross at any point, and (3) no two crossings can occur at the same horizontal level. The first two axioms can be satisfied by spreading strands out horizontally and the third axiom requires one crossing to be shifted up or down slightly in relation to another. In addition, each strand of a braid can only cross downward, they do not cross upward and double back on themselves. An example of a braid and a non-braid is in Figure 1.

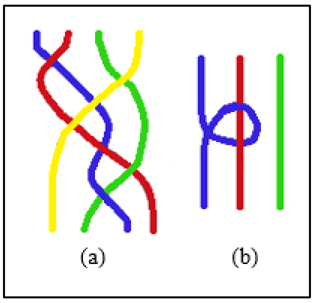


Figure 1 A Braid(a) and Non-Braid(b)

Generators of B_n consist of σ_i through σ_{n-1} and their inverses, including the identity braid. This symbol B_n is used to denote an n-strand braid where $n \geq 1$. For generators, symbols $(\sigma_i, \sigma_{i+1}, \sigma_{i+2}, \ldots, \sigma_{n-1})$ are used to denote the crossing of left strand in braids that cross under its adjacent strand, while the symbols $(\sigma_i{}^{-1}, \sigma_{i+1}{}^{-1}, \sigma_{i+2}{}^{-1}, \ldots, \sigma_{n-1}{}^{-1})$ or the inverses are used to denote the crossing of left strand in braids that cross over its adjacent strand [13][22].

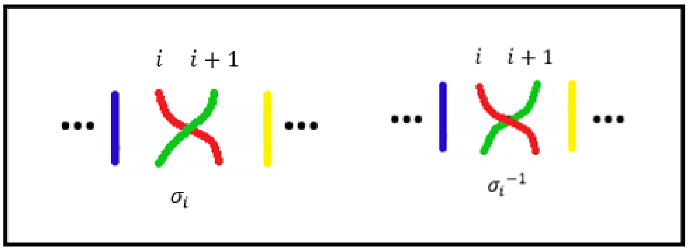


Figure 2 The Artin Generators

The following diagrams on the next page show the generators from B_2 to B_4.

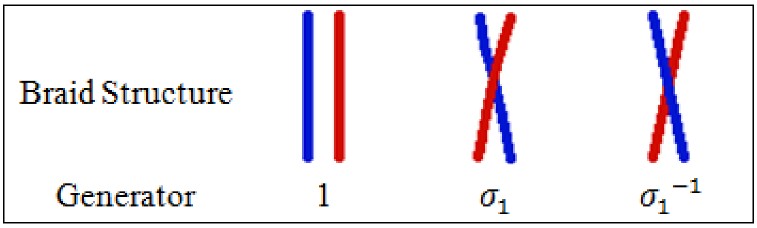


Figure 3 Generators of Braid in Two Strands(B_2)

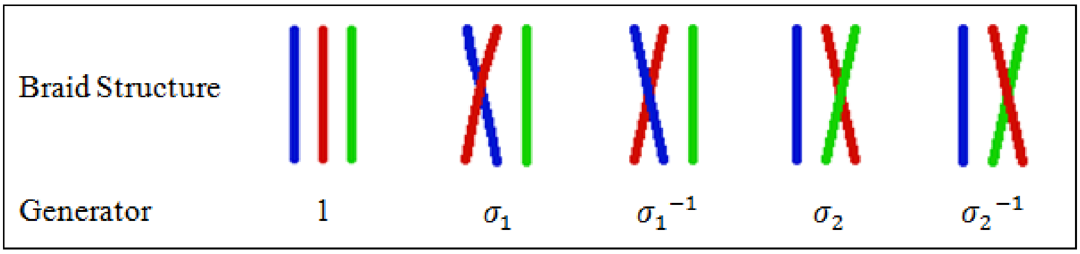


Figure 4 Generators of Braid in Three Strands(B_3)

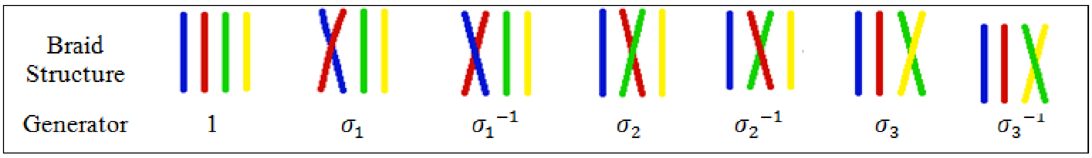


Figure 5 Generators of Braid in Four Strands(B_4)

Construction of braids can be made from left to right or from top to bottom. But the braid word is read from left to right only. Thus, figure 6 determines the words $\sigma_1\sigma_3\sigma_2\sigma_1\sigma_2{}^{-1}\sigma_3\sigma_2$ which are read: first σ_1 then $\sigma_3$then σ_2 then σ_1 then $\sigma_2{}^{-1}$ then σ_3 and σ_2.

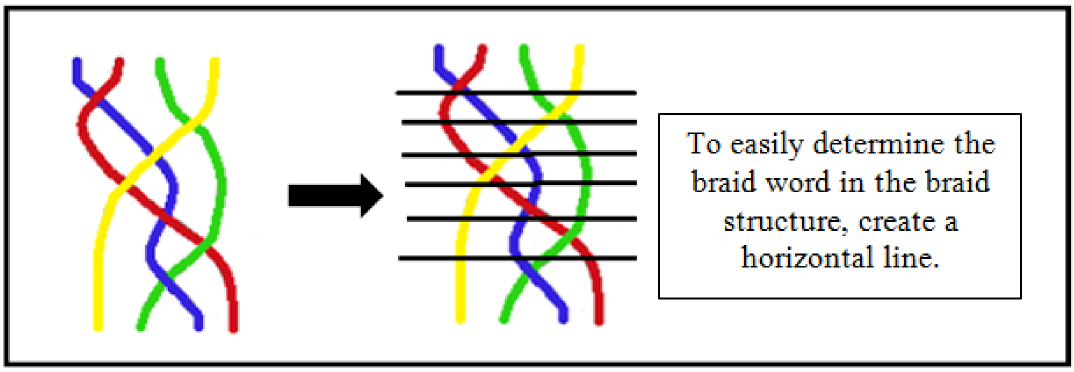


Figure 6 Braid in Four Strands with the Generator of $\sigma_1\sigma_3\sigma_2\sigma_1\sigma_2^{-1}\sigma_3\sigma_2$

In general, braids can have any number of strands. Numbering the strands from left to right, from $1, 2, 3 \ldots n$,the symbol (σ_i) is used for a positive crossing of the i^{th} strand with the $(i+1)^{th}$ strand and symbol (σ_i^{-1}) is used for a negative crossing between those strands (http://www.cmis.brighton.ac.uk/staff/agf/FinalBraidCrypto/ braidIntroduction2.htm).

Some existing properties of braid are presented in order to developed additional basic properties and propositions.

2. Objectives of the Study

This study aimed to:

1. Develop and prove other properties and propositions from existing properties of mathematical braids.
2. Generate an algorithm using MATLAB to generate the total number of generators and total number of crossings given particular number of strands.
3. Identify practical applications or relevance of mathematical braids to everyday life.

3. Methodology

This study used descriptive exploratory research. Exploratory research is a research conducted for a problem that has not been studied more clearly and aims to establish properties and develops operational definitions. Also, exploratory research improves the final research design. The researchers performed numerous exploration and investigation analysis on the existing properties of braid. New properties are developed together with their proof. This includes a broad knowledge of mathematical concepts in topology and MATLAB to accomplish the study. Furthermore, modulo art was utilized to further enhance the concept of mathematical braids. The number of crossings and number of generators can be used to create new designs using modular operation.

4. Results

4.1 Properties of Mathematical Braids

Definition 1. B_n is the symbol used to denote$n$-strand braid where n is the number of strand/s.

Definition 2. C_n is the symbol used to denote the number of crossing/s in n-strand braid where n is the number of strand/s.

Braid strands may cross over or cross under each other strand. This is simply called the crossing/s of braid. The first property is all about computing the total number of crossings in n-strand braid where$n \geq 1$.

Property 1. The total number of crossings in$n$-strand braid(B_n) can be solved using the formula $2n - 2$ where$n \geq 1$.

Proof

Observe that in every two strands of a braid, there are exactly two crossings. A strand crosses under its adjacent strand and another strand crosses over its adjacent strand.

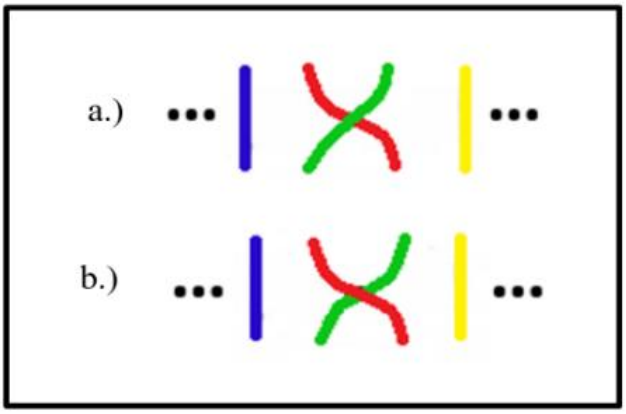


Figure 7 (a) Positive Crossing and (b) Negative Crossing of Braid

For B_n, where $n = 1$, no crossing occurs since no strand can cross over or cross under the adjacent strand.

For B_n, where $n = 3$, there exist two crossings for the first and second strands and two crossings again for the second and third strands. Therefore, there are exactly four crossings in 3 strand braids.

For B_n, where $n = 4$, there exist two crossings for the first and second strands, two crossings again for the second and third strands, and two crossings for the third and fourth strands. Therefore, there are exactly six crossings in 4 strand braids.

In general, the number of crossings can be solved using the formula $2n - 2$, where n is the number of strands. Observe the diagram below.

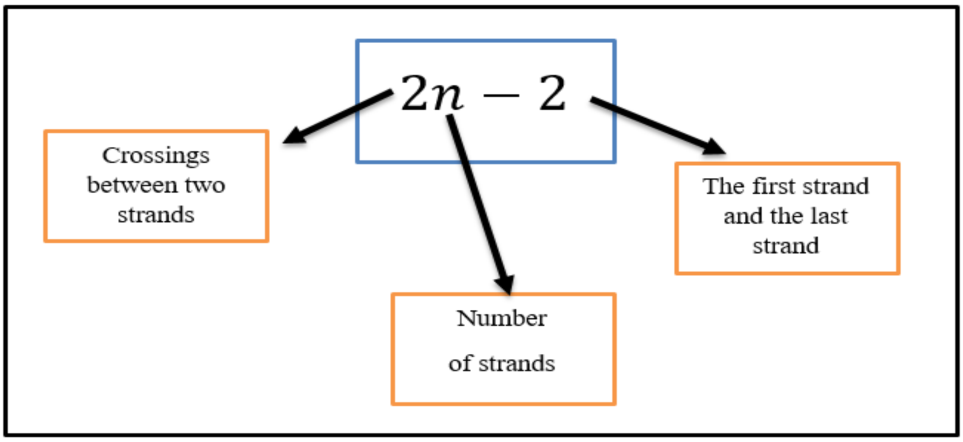


Figure 8 Formula of the Total Number of Crossings

From the figure on the previous page, there are two crossings in every two strands of a braid, the strandthat crosses over and the strand that crosses under. Subtract the first and last strand since there are no crossings between these strands and do not have an adjacent strand. To clearly understand this property, an example is given below.

For better understanding, other braid structures are given below. Begin with the first strand, followed by the second strand, until the last strand of the braid.

1. There are two crossings for the first and second strands

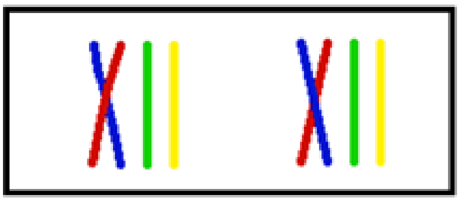


Figure 9 Crossings of the First and Second Strands

2. Two crossings for the second and third strands

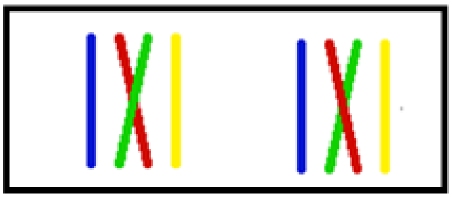


Figure 10 Crossings of the Second and Third Strands

3. Another two crossings for the third and fourth strands.

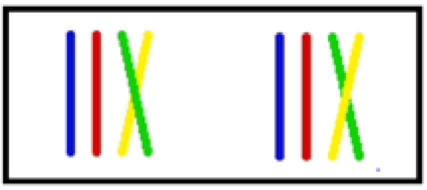


Figure 11 Crossings of the Third and Fourth Strands

Adding the three crossings from the previous page, six crossings are identified. Therefore, the total number of crossings in a four-strand braid (B_4) is six (6).

The next property is about the parity of the number of crossings in an n-strand braid.

Definition 3. Any number divisible by 2 is an even number and any number in the form $2n$ is also even, where *n* is an integer.

Property 2. The number of crossing/s in n-strand braid (B_n) is even, where $n \geq 1$.

Proof

Let C_n represent the total number of crossings such that $C_n = 2n - 2$. Observe the table below:

Table 1 Positive Numbers in Relation to Even Numbers

Positive Numbers(n)	Even numbers ($2n$)
1	*2*
2	*4*
3	*6*
4	*8*
5	*10*
…	…
$n - 1$	$(2n - 2)$
n	$2n$

Let E_n represent an even number, that is, $E_n = 2n$, where *n* is an integer. Then the sum of the first *n* values of E_i is:

$$\sum_{i=1}^{n} E_i = 2n = n^2 + n \quad (1)$$

$$2 + 4 + 6 + \cdots + 2n = n^2 + n$$

Table 2 Number of Crossing/s in Relation to Number of Strand/s

Number of Strand (n)	Number of Crossings ($2n - 2$)
1	*0*
2	*2*
3	*4*
4	*6*
5	*8*
…	…
n – 1	*(2n – 4)*
n	*(2n – 2)*

C_n represents the total number of crossings, that is$C_n = 2n - 2$, where $n \geq 1$. Then the sum of the first n values of C_i is:

$$\sum_{i=1}^{n} C_i = 2n - 2 = n^2 - n \quad (2)$$

$0 + 2 + 4 + 6 + \cdots + (2n - 2) = n^2 - n$ which could be proven by Mathematical Induction.

The crossing of left strand in braids that cross under its adjacent strand is also called the positive crossing of a braid while the crossing of left strand in braids that cross over its adjacent strand is the negative crossing of a braid. The next property is all about solving for the total number of generators n-strand braid (B_n) where $n \geq 1$.

Definition 4. Let G_n be the symbol used to denote the total number of generators/s in n-strand braid where n is the number of strand/s.

Definition 5. Let l_n be denoted as the position of the strand in the braid.

Property 3. The total number of generators in n-strand braid (B_n) can be solved using the formula $2n - 1$where $n \geq 1$.

Proof

In every generator of$n$-strand braid (B_n), there exists a set of $\{l_i, l_{i+1}, l_{i+2}, l_{i+3} \ldots l_{n-2}, l_{n-1}, l_n\}$ where l_i is denoted as the position of the strand in the braid. These strands can cross passing the left strand either under or over the right strand which is denoted by σ_i and $\sigma_i{}^{-1}$.

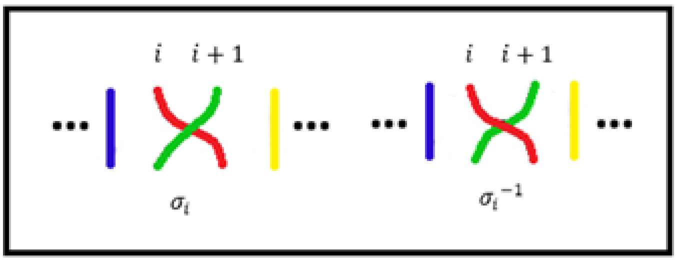


Figure 12 Illustration of σ_i and $\sigma_i{}^{-1}$

For every two adjacent strands, there exist two generators, σ_i and $\sigma_i{}^{-1}$. From existing properties *Property 1(Group)*, it is already proven that there exists an identity element(σ_e) in every n-strand braid. Thus, in every two strands, there are two generators and anidentity elementnamely; σ_1, $\sigma_1{}^{-1}$ and σ_e .

In getting the total number of generators, observe the table below.

Table 3 The Total Number of Generators/s in Relation to Number of Strand/s

Number of Strand (n)	Numbers of Crossings $(2n-2)+$ identity element (σ_e)
1	$0 +$ identity element (σ_e)
2	$2 +$ identity element (σ_e)
3	$4 +$ identity element (σ_e)
4	$6 +$ identity element (σ_e)
5	$8 +$ identity element (σ_e)
…	…
$n-1$	$(2n-4)$ $+$ identity element (σ_e)
n	$(2n-2) +$ identity element (σ_e)

As reflected from the table above, to get the total number of generators in n-strand braid (B_n), add the total number of crossings $(2n-2)$ and the identity element (σ_e). The "identity element" in each line of the table indicates $+1$ to the number of crossings. Thus

$$G_n = (2n-2) + 1 \tag{3}$$

Simplifying,

$$G_n = 2n - 1$$

Therefore, the formula for finding the total number of generators in n-strand braid (B_n) $is\ 2n - 1$. To clearly understand the property, an example is given below.

Example 2. How many number of generators are in 143 strand braid (B_{143})

Using the formula,

$$G_n = 2(n) - 1$$

By substituting n, it becomes

$$G_n = 2(143) - 1$$

By multiplication,

$$G_n = \ 286 - 1$$

It is then,

$$G_n = \ 285$$

Thus, the total number of generators in 143 strand braid is 285. The next property is similar with property 2. It is about the total number of generators in n-strand braid.

Definition 6. Any number not divisible by 2 is odd and of the form $2p + 1$.

Property 4. The total number of generators in n-strand braid (B_n) is odd number where $n \geq 1$.

Proof

Let C_n represent the total number of generators such that $G_n = 2n - 1$.

Observe the table on the next page;

Positive Numbers in Relation to Odd Numbers

Positive Numbers(n)	Odd numbers($2n+1$)
1	3
2	5
3	7
4	9
5	11
...	...
$n-1$	$(2n+2)$
n	$(2n+1)$

Let O_n represent an odd number, that is, $O_n = 2n+1$, where *n* is an integer. Then the sum of the first *n* values of O_i is;

$$\sum_{i=1}^{n} O_i = 2n+1 = (n^2+2n) \quad (4)$$

$$3+5+7+\cdots+(2n+1) = (n^2+2n)$$

Table 4 Number of Generator/s in Relation toNumber of Strand/s

Number of Strand (n)	Number of Generator/s ($2n-1$)
1	1
2	3
3	5
4	7
5	9
...	...
$n-1$	$(2n-1)$
n	$(2n+1)$

G_n represent the total number of generators such that $G_n = 2n-1$. Summation of the total number of generators:

$$\sum_{i=1}^{n} G_i = 2n-1 = n^2 \quad (5)$$

$$1+3+5+7+\cdots+(2n-1) = n^2$$

Which is correct by mathematical induction,

$$k^2+2k+1 = (k+1)(k+1)$$

$$k^2+2k+1 = k^2+2k+1 \rightarrow \text{True}$$

Hence, $\sum_{i=1}^{k} G_i = 2k-1 = k^2$ is odd.

5. The Connections of Mathematical Braids to Exponentiation Theorems

The construction of braid involves the operation called the concatenation. In concatenating braids, the researchers found out propositions that are relatedto exponential theorems which are very helpful in concatenating braids.

Theorem 1. (Exponential theorem).$a^0 = 1$

In exponential theorem ($a^0 = 1$) if a is any integer number raise to zero the answer is always equal to 1 while the proposition 1 in mathematical braids (${\sigma_i}^0 \neq 1$) for $\sigma_i$is any element of n-strand braid (B_n) raise to zero is not equal to identity element which is σ_e or 1.

For example,

In Exponential theorem, if = 3, then $3^0 = 1$ while in mathematical braids if $\sigma_i = \sigma_3$, then ${\sigma_3}^0 = \sigma_3$. Thus, *Proposition 1* contradicts *Theorem 1*-Exponential Theorems.

Proposition 1. Any positive integer $r$such that ${\sigma_i}^r = 1$does not exist in n-strand braid

(B_n) where $r \geq 0$.

Proof

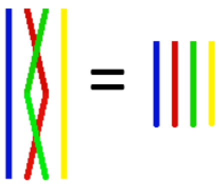
Consider that ${\sigma_i}^r = 1$with a positive integer r for $\sigma_i \in B_n$.

By *Definition 1* from existing properties,

$\sigma_i * {\sigma_i}^{-1} = 1$

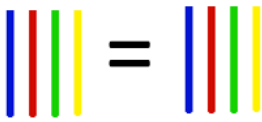
By concatenation,

$$\sigma_i {\sigma_i}^{-1} = 1$$



Simplifying using handle reduction, it becomes

$$1 = 1$$

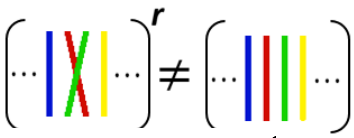


From the braid structure, one can simply understand that in getting the identity element, there must be two given generators (1) any generator and (2) the inverse of the given generator.

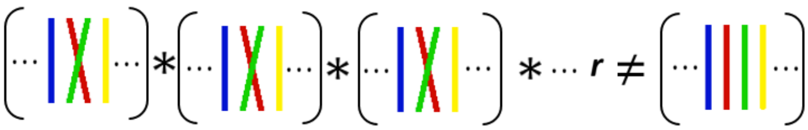
Further, by concatenation, the identity element could be taken. Therefore, concatenation of the same generators σ_i will not be equal to 1, which is

$$\sigma_i{}^r \neq 1$$

Simplifying, it becomes

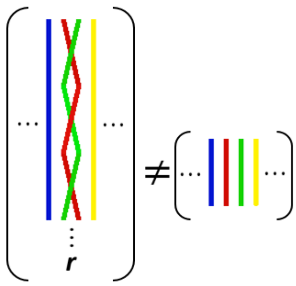


$$\sigma_i * \sigma_i * \sigma_i * \cdots * r \neq 1$$



By concatenation, it results to

$$\sigma_i * \sigma_i * \sigma_i * \cdots * r \neq 1$$



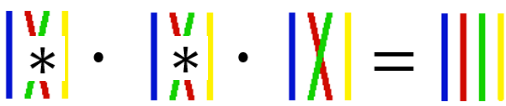
Therefore, $\sigma_i{}^r \neq 1$.

To clearly understand the proposition, an example and a braid structure is given below to show that $\sigma_i{}^r \neq 1$.

Example 1. Let $B_n = B_4$, $r = 3$, $\sigma_i = \sigma_2$ and $\sigma_e = 1$.

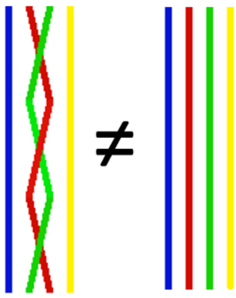
This is illustrated as

$$\sigma_2{}^3 \neq 1$$



By concatenation of braid

$$\sigma_2\sigma_2\sigma_2 \neq 1$$



Using handle reduction, pushing, pulling, and moving, the braids from the left side will not go back to its original position.

Theorem 2. (Exponential theorem).$(ab)^r = a^r b^r$

In exponential theorem ($(ab)^r = a^r b^r$), if a and b are any positive integer raise to positive integer r then it is equal to $a$raise to positive integer r multiplied to b raise to positive integer r. While in mathematical braids $\left(\sigma_i * \sigma_j\right)^r \neq ({\sigma_i}^r * {\sigma_j}^r)$ for $\sigma_i$and $\sigma_j$are any element in n-strand braidraise to positive integer r then it is not equal to $\sigma_i$raise to positive integer r concatenated to σ_j raise to positive integer r.

For example,
In Exponential theorem, if $a = 7$, $b = 5$ and $r = 6$ then$(7x5)^6 = 7^6x5^6$ while in mathematical braids, if$\sigma_i = \sigma_7$, $\sigma_j = \sigma_5$ and $r = 6$ then $(\sigma_7 * \sigma_5)^6 \neq {\sigma_7}^6 * {\sigma_5}^6$. Thus, ***Proposition 2*** contradicts *Theorem 2*-Exponential Theorems

6. Summary

1. These are additional basic properties found out by the researchers:
   a. The total number of crossings in$n$-strand braid(B_n) can be solved using the formula $2n - 2$where$n \geq 1$.
   b. The number of crossing/s in$n$-strand braid(B_n) is even, where $n \geq 1$.
   c. The total number of generators in n-strand braid(B_n) can be solved using the formula$2n - 1$where$n \geq 1$.
   d. The total number of generators in$n$-strand braid$(B_n)$is odd number where $n \geq 1$.

   The following are the propositions found to have connections of exponential theorems to mathematical braids:
   a. Any positive integer $r$such that ${\sigma_i}^r = 1$does not exist in n-strand braid$(B_n)$where $r \geq 0$.
   b. For any positive integer r, then$\left(\sigma_i * \sigma_j\right)^r \neq {\sigma_i}^r * {\sigma_j}^r$.

c. If an n-strand braid(B_n)has an exactly one generator$\sigma_i \in B_n$, then ${\sigma_i}^{r+s} = {\sigma_i}^r * {\sigma_i}^s$, for any positive integer *r* and *s*.

d. If an n-strand braid(B_n) has an exactly one generator $\sigma_i \in B_n$, then ${\sigma_i}^{rs} = ({\sigma_i}^r)^s$, for any positive integer *r* and *s*.

e. If $\sigma_i, \sigma_j \in B_n$, then $(\sigma_i * \sigma_j)^{-1} \neq (\sigma_i)^{-1} * (\sigma_j)^{-1}$.

f. If $\sigma_i \in B_n$, then$({\sigma_i}^{-1})^{-1} = \sigma_i$.

g. If an n-strand braid(B_n)has an exactly one generator$\sigma_i \in B_n$, then

h. $({\sigma_i}^r)^{-1} = ({\sigma_i}^{-1})^r$, for any positive integer *r*.

2. MATLAB was utilized to generate an algorithm to output the total number of crossings and total number of generators given a particular number of strands.
3. A modular design can be generated using the number of generators and number of crossings.

7. Conclusions

After an extensive research on mathematical braids, these are the conclusions of the study:

1. The additional basic properties proven are (i) the total number of crossings in$n$-strand braid (B_n) can be solved using the formula $2n - 2$ where $n \geq 1$, (ii) the number of crossing/s in n-strand braid (B_n) is even, where $n \geq 1$, (iii) the total number of generators in n-strand braid (B_n) can be solved using the formula$2n - 1$ where $n \geq 1$, (iv) the total number of generators in n-strand braid (B_n) is odd number where $n \geq 1$, with nine corresponding propositions that involve the connections of exponential theorems to mathematical braids.
2. Using MATLAB, the algorithm used to generate the total number of crossings and total number of generators given any particular number of strands are (i) insert any number of strand in the program, and (ii) the program will now compute the total number of crossings and total number of generators.
3. Braids are applications of mathematical concepts where generators are identified and using modular operation, the number of crossings, and number of generators can be used to create new modular designs.

8. Recommendations

After undertaking a thorough study of mathematical braids, the following are recommended:

1. Other properties of mathematical braids may be considered.
2. Develop a program on mathematical braids that could generate braid design using the generators of braid.

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