RELATIONS OF LEMOINE CIRCLE WITH A SYMMEDIAN POINT

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Abstract. Any triangle $ABC$ have three symmedian lines that intersect at one point $K$ that called symmedian point, so triangle $ABC$ can be partitioned into three triangles, i.e. triangle $ABK$, triangle $ACK$ and triangle $BCK$. From each of the triangle can be constructed the circumcircle, for example with a center point $R$, $Q$ and $P$ respectively. If these points linked, so will form a triangle and has a centroid point, say point $M$. In addition, each of the circumcircle triangle $ABK$, triangle $ACK$ and triangle $BCK$ will intersect in six points on the sides and the extension side triangle $ABC$ which will be lie on one circle that known as third Lemoine circle. If $L$ is the center point of the third Lemoine circle, then in this article discuss that $K$, $L$ and $M$ are collinear.

1. INTRODUCTION

Triangle $ABC$ have any special line symmedian [11] which intersect at one point called the point symmedian [1] which is denoted by $K$. So triangle $ABC$ can be partitioned into three triangles, i.e. triangle $BCK$, triangle $ACK$ and triangle $ABK$. In each triangle can be constructed circumcircle with the center point $P$, $Q$ and $R$, respectively. If the points $P$, $Q$ and $R$ are linked by a line segment, it will form a triangle $PQR$. In addition, from constructing three circumcircle of triangle $BCK$, triangle $ACK$ and triangle $ABK$, will form the intersection point of the circle with the sides and the extension side triangle $ABC$ on six points. If the sixth point of
intersection is connected, it will form a circle known as the third Lemoine circle [6]. In addition to the third Lemoine circle, there are the first Lemoine circle and second Lemoine circle [11, 12, 13, 14, 15]. The first Lemoine circle constructed by drawing three parallel lines to the sides of the triangle where all three lines through point symmedian $K$ so it will intersect with the sides of the triangle at six points. Then the sixth point will be lie the one circle. While the second Lemoine circle constructed by drawing three anti parallel lines to the sides of the triangle are through the symmedian point so it will intersect in six points with a side of the triangle. Then the sixth point will be lie the one circle.

This article will discuss the relationship of third Lemoine circle, symmedian point of triangle ABC and centroid point of triangle $PQR$. If $M$ is the centroid point [8] of triangle $PQR$, and $L$ is the center point of the third Lemoine circle while $K$ is the symmedian point of triangle $ABC$ so in this article will be proven that the three points $K$, $L$ and $M$ are collinear.

2. SOME BASIC CONCEPTS

Any triangle $ABC$ has three median line [9] which, according to Ceva’s Theorem [10] will intersect at one point called the centroid point [8]. In addition to the median line, there are internal angle bisector also on a triangle, then if the A-median line reflected in the A-internal angle bisector will produce a line known as the symmedian line [11]. In any triangle, three symmedian lines will intersect at one point called the symmedian point [1], and denoted by $K$.

**Definition 2.1** (Symmedian line). In a triangle $ABC$, the reflection of the A-median in the A-internal angle bisector is called the A-symmedian of triangle $ABC$. Similarly, we can define the B-symmedian and the C-symmedian of the triangle. Considering Figure 1, $AM$ is median-A, $AD$ and bisector angle-A. If $AM$ is reflected in $AD$, so we have $AX$. The line $AX$ is called symmedian-A.

A circle can be constructed from a triangle. For example is circumcircle [7] with the point center is circumcenter, incircle triangle, and the tangent circle of triangle [6]. A circle that constructed from symmedian point is called Lemoine circle. L. Sammy and P. Cosmin divide the Lemoine circle in two forms, i.e. first Lemoine circle and second Lemoine circle. Grinberg divide the Lemoine circle in three forms are first Lemoine circle, second
Teorema 2.2 (The first Lemoine Circle). Let $K$ be the symmedian point of the triangle $ABC$ and let $x; y; z$ this time be the parallels drawn through $K$ to $BC$, $CA$, and $AB$, respectively. Prove that the six points determined by $x$, $y$, $z$ on the sides of $ABC$ all lie on one circle.

Proof. See [13].

Teorema 2.3 (The Second Lemoine Circle). Let $K$ be the symmedian point
of triangle $ABC$ and let $x$, $y$, $z$ be the antiparallels drawn through $K$ to the lines $BC$, $CA$, and $AB$, respectively. Prove that the six points determined by $x$; $y$; $z$ on the sides of $ABC$ all lie on one circle. This is called the First Lemoine Circle of triangle $ABC$.

**Figure 3:** Second Lemoine Circle.

**Proof.** See [13].

**Theorem 2.4** (The Third Lemoine Circle) Let the circumcircle of triangle $BLC$ meet the lines $CA$ and $AB$ at the points $Ab$ and $Ac$ (apart from $C$ and $B$). Let the circumcircle of triangle $CLA$ meet the lines $AB$ and $BC$ at the point $Bc$ and $Ba$ (apart from $A$ and $C$). Let the circumcircle of triangle $ALB$ meet the lines $BC$ and $CA$ at the points $Ca$ and $Cb$ (apart from $B$ and $A$). Then, the six points $Ab$; $Ac$; $Bc$; $Ba$; $Ca$; $Cb$ lie on one circle. This circle is a Tucker circle, and its midpoint $M$ lies on the line $UL$ and satisfies $LM = \frac{1}{2}LU$ (where the segments are directed). The radius of this circle is $\frac{1}{2} \sqrt{9r_1^2 + r^2}$, where $r$ is the circumradius and $r_1$ is the radius of the second Lemoine circle of triangle $ABC$.

**Proof.** See [6].

**Theorem 2.4** (Centroid Theorem) If $D$, $E$ and $F$ respectively are the midpoints of the sides $BC$, $CA$ and $AB$ at a triangle $ABC$, then the line $AD$, $BE$ and $CF$ are concurrent at point $G$ and
In this article, the relationship of Lemoine circle that will be discussed is the relationship the third Lemoine circle with symmedian point of triangle $ABC$ and centroid point of triangle $PQR$ that formed by connecting the center point of the circumcircle of triangle $AKB$, triangle $BKC$ and triangle $AKC$. If $L$ the point center of third Lemoine circle, $K$ symmedian point of triangle $ABC$, and $M$ is centroid point $PQR$, will be shown that the $K$, $L$ and $M$ are collinear.

**Theorem 3.1** Let the symmedian point of triangle $ABC$ is $K$, the center point of third Lemoine circle is $L$ and $M$ is the centroid of triangle $PQR$, then the points $K$, $L$ and $M$ are collinear.
Figure 5: Median line and centroid point.

Figure 6: Points $K$, $L$ and $M$. 
Proof $K$ is the centroid point of triangle $B_aBB_c$, and $B_m$ is the point of intersection of the median line of of triangle $B_aBB_c$ on the side $B_cB_a$ and $T$ is the point of intersection of the median-$B$ line with triangle $PQR$, on the $PR$ side, as shown in Figure 7.

![Figure 7: Triangle $B_aBB_c$ with centroid $K$.](image)

Then by Theorem 2.4 it is shown that

$$KB_m = \frac{1}{2}BK$$

Then by connecting the points $P$ and $R$ with symmedian point and point $B$ as shown in Figure 8. In order to obtain that the quadrilateral $KRBP$ is a kite with two diagonal cut-off point at the point $T$, so

$$TK = \frac{1}{2}BK = KB_m$$

Then make a point $S$ such that $LT \parallel SB$ as shown in Figure 9.

Based on the theorem 2.4 shows that

$$KL = \frac{1}{2}KU,$$
Figure 8: Triangle $B_aBB_c$ with centroid $K$.

Figure 9: Point $K$, $L$, $M$, $U$ and $S$. 
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\[ \frac{KL}{KS} = \frac{KL}{KU} = \frac{1}{2}. \]

By drawing the line \( UN \) as shown in Figure 9, we obtain \( \Delta KSB \sim \Delta KUN \) and by applying

\[ \frac{KU}{UN} = \frac{KS}{SB} = \frac{KL}{LT} \]

Since the point \( Q, K \) and \( N \) are collinear, then the point \( S, K \) and \( U \) are collinear too. By painting a line parallel to the \( UN \) through \( B_m \) there will be a cut-off point on the segment of the \( UN \) in point \( H \) and by applying

\[ KH = \frac{1}{2} KU \]

With the enactment of this comparison, then the point \( K, H \) and \( U \) are collinear. In other words, \( L, K \) and \( H \) are collinear. Furthermore, \( M \) is the centroid of triangle \( PQR \) with \( J \) is median-\( Q \), by drawing a line that through the \( U \) and parallel to the line \( KJ \) as in Figure 10.

Figure 10: Triangle \( JVX \).

so

\[ \frac{KJ}{XV} = \frac{KM}{KJ} = \frac{KM}{KU} = 1/2 \]
\[ KM = KU = KH \]

With proven that \( KM = KH \), it states that the point \( H = M \). Thus concluded that the \( K, L \) and \( M \) are collinear.

4. CONCLUSION

The point center of third Lemoine circle of triangle \( ABC \), symmedian point of \( ABC \) and the centroid point of triangle \( PQR \) are collinear.

References


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