Bulletin of Mathematics
Vol. 10, No. 02 (2018), pp. 157–165
ISSN Printed: 2087-5126; Online: 2355-8202
https://talenta.usu.ac.id/index.php/bullmath

AN ANALYSIS OF THE RELATIONSHIPS OF k-TRIBONACCI NUMBERS IN MODULO 6

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Abstract. *This article analysed the relationship of k-Tribonacci in modulo 6 by using a method practiced by Muslim [International Mathematical Forum, 13 (2018), 225-231]. k-Tribonacci numbers in Modulo 6 are differentiated into 4 special cases namely* $k \equiv 0 \bmod 6$, $k \equiv 1 \bmod 6$ *and* $k \equiv 4 \bmod 6$, $k \equiv 0 \bmod 6$ *of which p is a prime number.*

1. INTRODUCTION

Nowadays various researches on Fibonacci numbers has developed.The sequence of Fibonacci number is expressed as $0, 1, 1, 2, 3, 5, 8, 13, \ldots$. The number after in the number petten is generated by adding the previous two numbers. The number found by Leonardo da Pisa has a formula $F_{n+1} = F_n + F_{n-1}$ with $F_0 = 0$, and $F_1 = 1$. Generalitation of Fibonacci number has been found in various article and books. Research on Fibonacci numbers has developed to Fibonacci numbers in modulo n. Researches on Fibonacci numbers in modulo n were conducted by Vince [10] and Andreassian [1] discussing identities or properties of Fibonacci numbers in modulo n. Then, Ehrlich [3] discussed about period of Fibonacci numbers in modulo n.

Received 23-12-2018, Accepted 29-12-2018.
2017 Mathematics Subject Classification: 11B39,
Key words and Phrases: k-tribonacci numbers, k-tribonacci numbers identity, numbers in modulo 6

Tribonacci numbers are development of Fibonacci numbers. Koshy [9] said that Tribonacci numbers were initially learned in 1963 by Feinberg. Tribonacci numbers begins with first term which is 0, the second term is also 0 and its third term is 1. Then, each subsequent term is obtained by adding three previous terms. Devbhadra[2] Tribonacci sequence consists of terms $T_1, T_2, T_3, T_4, \ldots$, where we define $T_1 = 0, T_2 = 1, T_3 = 1$ and any following terms given by $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, for $n \geq 1$. Fist few Tribonacci number are $0, 1, 1, 2, 4, 7, 13, \ldots$.

In 2018, Muslim [8] conducted a research on properties or identities of Fibonacci and Tribonacci numbers in modulo 6. The reseach shows that there is a relationship between Fibonacci and Tribonacci numbers in modulo 6 and Fibonacci and Tribonacci numbers in modulo 2 and modulo 3.

k-Fibonacci numbers are also development of Fibonacci numbers. k-Fibonacci numbers also begins with 0 and 1 but subsequent number depends on specified value of k. k-Fibonacci number denote by $F_{k,n+1} = kF_n + F_{n-1}$. k-Fibonacci numbers were put forward by S. Falcon and A. Plaza [4] from recursive learning of two geometric transformation which is well known as partition of *four-triangle longest-edge* (*4TLE*).

In 2018, Wahyuni [11] conducted a research on identities of k-Fibonacci numbers in modulo m and modulo 10. In certain moduli, some identities were found such as the relationship between k-Fibonacci numbers in modulo 6 and k-Fibonacci numbers in modulo 2 and 3, the relationship between k-Fibonacci numbers in modulo 10 and k-Fibonacci numbers in modulo 2 and 5 and other properties as well.

Based on the background of the problem, thus the researcher was interested to define a number namely k-Tribonacci and follow the method conducted by Muslim [8] in order to obtain a pattern of k-Tribonacci numbers in modulo 6. Then, there were analyses of the relationship between their terms and the relationship between k-Tribonacci in modulo 6 and k-Tribonacci in factor modulo 6.

2. k-TRIBONACCI NUMBERS IN MODULO 6

Lather [5] denote by $F_{k,n}$ the n^{th} k-Tribonacci number, where $F_{k,n} = kF_{k,n-1} + F_{k,n-2} + F_{k,n-3}$, for $n \geq 3$ with initial conditions $F_{k,0} = 0, F_{k,1} = 1, F_{k,2} = 1$. By referring to the definitions of Fibonacci and k-Fibonacci numbers thus in this article k-Tribonacci is defined as follow:

Definition 1.1 *k-Tribonacci numbers are numbers whose definition is:*

$$T_{k,n+1} = k^2 T_{k,n-2} + kT_{k,n-1} + T_{k,n},$$

for $n \geq 3$ with $T_0 = 0, T_1 = 1$, and $T_2 = 1$

By using the definition of k-Tribonacci numbers its value is calculated and its congruence is determined in modulo 6. The symbol of k-Tribonacci numbers is $^*T^6_{k,n}$ and their values are available in the following Table 1.

From Table 1 it is obvious that the values of k-Tribonacci numbers in modulo 6 return to their original values in 4 different cases. First, if $k \equiv 0 \bmod 6$ thus values of k-Tribonacci numbers recur after the second term whose values are 1 for all of $n > 2$. Second, if $k \equiv 1 \bmod 6$ thus values of k-Tribonacci numbers recur after the fifty second term. Third, if $k \equiv p \bmod 6$ of which p is prime numbers less or equal to 5, thus values of k-Tribonacci numbers recur after the fourth term. Fourth, if $k \equiv 4 \bmod 6$, thus values of k-Tribonacci numbers recur after the thirteenth term.

3. PATTERN OF k-TRIBONACCI NUMBERS IN MODULO 6

a. Case of $k \equiv 0 \bmod 6$
The value of k-Tribonacci numbers in modulo 6 for $k \equiv 0 \bmod 6$ is 1 for each term other than their initial term, therefore it is not really interesting in determining their patterns.

b. Case of $k \equiv 1 \bmod 6$
The value of k-Tribonacci numbers in modulo 6 for $k \equiv 1 \bmod 6$ recurs after the 52^{th} term, in other word k-Tribonacci numbers in modulo 6 have the order of 52. The recurrence of values of k-Tribonacci in modulo 6 after the 52^{th} term could be patterned by using the following letters

$$A_6B_6C_6D_6A_6B_6E_6F_6G_6H_6I_6J_6G_6,$$

of which A_6 recurs 1, 1, 4, 4, B_6 recurs 1, 4, 4, 1, C_6 recurs 2, 2, 5, 5, D_6 recurs 4, 1, 1, 4, E_6 recurs 0, 0, 3, 3, F_6 recurs 2, 5, 5, 2, G_6 recurs 3, 3, 0, 0, H_6 recurs 5, 2, 2, 5, I_6 recurs 4, 4, 1, 1, and J_6 recurs 0, 3, 3, 0.

c. Case of $k \equiv p \bmod 6$
Just like the case of $k \equiv 1 \bmod 6$, in the case of $k \equiv p \bmod 6$ the recurrence happens after the fourth term, that is k-tribonacci numbers in

Table 1: k-Tribonacci numbers in modulo 6

n	${}^*T^6_{1,n}$	${}^*T^6_{2,n}$	${}^*T^6_{3,n}$	${}^*T^6_{4,n}$	${}^*T^6_{5,n}$	${}^*T^6_{6,n}$
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	1	1	1	1	1	1
3	2	3	4	5	0	1
4	4	3	4	1	0	1
5	1	1	1	1	1	1
6	1	1	1	1	1	1
7	0	3	4	3	0	1
8	2	3	4	5	0	1
9	3	1	1	3	1	1
10	5	1	1	5	1	1
11	4	3	4	1	0	1
12	0	3	4	3	0	1
13	3	1	1	3	1	1
14	1	1	1	1	1	1
15	4	3	4	1	0	1
16	2	3	4	5	0	1
17	1	1	1	1	1	1
18	1	1	1	1	1	1
19	4	3	4	1	0	1
20	0	3	4	3	0	1
21	5	1	1	5	1	1
22	3	1	1	3	1	1
23	2	3	4	5	0	1
24	4	3	4	1	0	1
25	3	1	1	3	1	1
26	3	1	1	3	1	1
27	3	1	1	3	1	1
28	4	3	4	1	0	1
29	4	3	4	1	0	1
30	5	1	1	5	1	1
31	1	1	1	1	1	1
32	4	3	4	1	0	1
33	4	3	4	1	0	1
34	3	1	1	3	1	1
35	5	1	1	5	1	1

modulo 6 for $k \equiv p \bmod 6$ have the order of 4. The recurrence of values could be also patterned as follow:

$$1_6 1_6 (k+1)_6 (k+1)_6$$

The pattern recurs after the fourth term and their values depend on their

n	$^*T^6_{1,n}$	$^*T^6_{2,n}$	$^*T^6_{3,n}$	$^*T^6_{4,n}$	$^*T^6_{5,n}$	$^*T^6_{6,n}$
36	0	3	4	3	0	1
37	2	3	4	5	0	1
38	1	1	1	1	1	1
39	3	1	1	3	1	1
40	0	3	4	3	0	1
41	4	3	4	1	0	1
42	1	1	1	1	1	1
43	5	1	1	5	1	1
44	4	3	4	1	0	1
45	4	3	4	1	0	1
46	1	1	1	1	1	1
47	3	1	1	3	1	1
48	2	3	4	5	0	1
49	0	3	4	3	0	1
50	5	1	1	5	1	1
51	1	1	1	1	1	1

k values.

d. Case of $k \equiv 4 \bmod 6$
In the case of $k \equiv 4 \bmod 6$ there is no interesting pattern in terms of letter symbols, however k-tribonacci numbers in modulo 6 for $k \equiv 4 \bmod 6$ have recurrence of term after the 13^{th} term. Those recurrence of values could be patterned as follow:

$$1_6 1_6 5_6 1_6 1_6 1_6 3_6 5_6 3_6 5_6 1_6 3_6 3_6$$

4. AN ANALYSIS OF THE RELATIONSHIP OF k-TRIBONACCI NUMBERS IN MODULO 6

Analysis of the relationship of k-tribonacci numbers in modulo 6 is an analysis of relationship of terms of k-tribonacci numbers whose congruence is determined in modulo 6. Some relationships obtained are explained in the following conjectures:

Conjecture 1.1 *For all n natural number, thus prevails*

$$^*T^6_{2 \bmod 6,n}\,{}^*T^6_{3 \bmod 6,n} \equiv {}^* T^6_{5 \bmod 6,n}$$

Proof: k-tribonacci numbers in modulo 6 for $k \equiv p \bmod 6$ have the order of 4 that is their terms recur after the 52^{th} and its multiple, therefore in order to show those conjectures it is made available Table 3.1, congruence of conjecture 1 in modulo 6.

Table 2: Congruence of conjecture 1.1

n	${}^{*}T^{6}_{2 \bmod 6,n}$	${}^{*}T^{6}_{3 \bmod 6,n}$	${}^{*}T^{6}_{2 \bmod 6,n}\,{}^{*}T^{6}_{3 \bmod 6,n}$	${}^{*}T^{6}_{5 \bmod 6,n}$
1	1	1	1	1
2	1	1	1	1
3	3	4	0	0
4	3	4	0	0
5	1	1	1	1
6	1	1	1	1
7	3	4	0	0
8	3	4	0	0

Conjecture 1.2 *For all n natural number, thus prevails*

$$\left(\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{2} + \left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{3}\right) \equiv 0 \bmod 6$$

Proof: Just like the proof of first conjecture, in second conjecture it is made available a table of congruence $\left(\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{2} + \left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{3}\right) \equiv 0 \bmod 6$ i.e. in the following Tabel 3.

Conjecture 1.3 *For all n natural number, thus prevails*

$$3\begin{vmatrix} {}^{*}T^{6}_{4 \bmod 6,n} & {}^{*}T^{6}_{4 \bmod 6,n+2} \\ {}^{*}T^{6}_{4 \bmod 6,n+3} & {}^{*}T^{6}_{4 \bmod 6,n+1} \end{vmatrix} \equiv 0 \bmod 6$$

Proof: Just like the first and second conjectures, in order to show the third conjecture it is made available a table of congruence $3\begin{vmatrix} {}^{*}T^{6}_{4 \bmod 6,n} & {}^{*}T^{6}_{4 \bmod 6,n+2} \\ {}^{*}T^{6}_{4 \bmod 6,n+3} & {}^{*}T^{6}_{4 \bmod 6,n+1} \end{vmatrix} \equiv 0 \bmod 6$ that is in Table 4.

Table 3: Congruence of conjecture 1.2

n	${}^{*}T^{6}_{4 \bmod 6,n}$	$\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^2$	$\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^3$	$\left(\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^2 + \left({}^{*}T^{6}_{4 \bmod 6,n}\right)^3\right)$
1	1	1	1	0
2	1	1	1	0
3	0	1	5	0
4	0	1	1	0
5	1	1	1	0
6	1	1	1	0
7	0	3	3	0
8	0	1	5	0
9	1	3	3	0
10	1	1	5	0
11	0	1	1	0
12	0	3	3	0
13	1	3	3	0

Table 4: Congruence of conjecture 1.3

n	${}^{*}T^{6}_{4 \bmod 6,n}$	${}^{*}T^{6}_{4 \bmod 6,n+1}$	${}^{*}T^{6}_{4 \bmod 6,n+2}$	${}^{*}T^{6}_{4 \bmod 6,n+3}$	$3\begin{vmatrix} {}^{*}T^{6}_{4 \bmod 6,n} & {}^{*}T^{6}_{4 \bmod 6,n+2} \\ {}^{*}T^{6}_{4 \bmod 6,n+3} & {}^{*}T^{6}_{4 \bmod 6,n+1} \end{vmatrix}$
1	1	1	5	1	0
2	1	5	1	1	0
3	5	1	1	1	0
4	1	1	1	3	0
5	1	1	3	5	0
6	1	3	5	3	0
7	3	5	3	5	0
8	5	3	5	1	0
9	3	5	1	3	0
10	5	1	3	3	0
11	1	3	3	1	0
12	3	3	1	1	0
13	3	1	1	5	0

5. CONCLUSION

The values of k-tribonacci numbers in modulo 6 recur in four different cases which are determined by their k values. By conducting analysis of the values of the terms, some relationships between terms of k-tribonacci numbers in modulo 6 were obtained namely:

For all n natural number, thus prevails

$$^{*}T^{6}_{2 \bmod 6,n}\,{}^{*}T^{6}_{3 \bmod 6,n} \equiv^{*} T^{6}_{5 \bmod 6,n},$$
$$\left(\left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{2} + \left({}^{*}T^{6}_{4 \bmod 6,n}\right)^{3}\right) \equiv 0 \bmod 6$$

and

$$3\begin{vmatrix} {}^{*}T^{6}_{4 \bmod 6,n} & {}^{*}T^{6}_{4 \bmod 6,n+2} \\ {}^{*}T^{6}_{4 \bmod 6,n+3} & {}^{*}T^{6}_{4 \bmod 6,n+1} \end{vmatrix} \equiv 0 \bmod 6$$

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