CONSERVATION AND DECAY IN PHYSICAL MODELLING

Frits P.H. van Beckum

Abstract. Mathematical modeling of physical phenomena is mainly governed by conservation laws (mass, momentum, heat, etc) and by principles like “Physics will always decay to equilibrium, will strive for minimal energy”, etc. Therefore, in the resulting mathematical model, e.g. a system of ordinary or partial differential equations, it makes sense to recognize and distinguish the terms and structures that are related to conservation from those that are responsible for decay. Examples are mainly taken in the field of heat transfer and wave propagation. Here, with “terms” we mean: the spatial derivatives of first and higher order and as examples of “structure” we first have the well-known Hamiltonian structure [1, 2](both in ordinary and in partial differential equations) which guarantees conservation, but we will also see its extension to models where decay plays a role. These insights can be of profit for numerical calculation [3]. In general purpose algorithms the conservation properties are generally lost, due to numerical errors. However, if we construct an algorithm following the special structure of the equations, then it will inherit the conservation properties. One of the consequences is e.g. that in models with decay, we can be sure that the decay that we calculate is not due to numerics, but is exactly representing the physics that we have modeled in the equations.

1. INTRODUCTION

Mathematicians will not aiming at a full definition agree that mathematics can be seen as a quantitative science about abstract objects and their properties. In university courses and textbooks mathematics may mainly

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seem a science of definition theorem proof, while in high school the memorizing of formulas and the application of tricks may sometimes dominate the perception rather than the understanding of methods and results.

Among these and more points of view, an important way of looking at mathematics is given by: a science of structures. This of course includes geometrical constructions but also algebraic structures like rings and modules, analytic structures like Banach spaces, and many others in discrete mathematics, probability, etc. In this paper we restrict to structures in partial differential equations, in particular the basic wave equations that are used in fluid dynamics and hydraulic engineering, and we give three examples of what these structures tell about the behavior of the solution, the propagation of physical quantities, and their conservation or decay.

The subjects collected here have been studied in cooperation projects between Indonesia and the Netherlands, which started in 1990 and are still going on. It is with great pleasure that I review some of them here in Medan.

2. RECOGNIZE THE EFFECT OF SPATIAL DERIVATIVES

This chapter will be about waves of the form \( u(x, t) \), function of time \( t \) and of one spatial variable \( x \). Specifically, we will consider waves that initially are harmonic, i.e. have a pure sinusoidal form, and we will consider how they deform as a solution of a linear partial differential equation of a specific form: first order in time and any finite (but still low) order in \( x \). At the end we will consider one nonlinear case, involving the most frequently occurring nonlinear term \( uu_x \).

**Translation equation**

The simplest partial differential equation one can think of is

\[
  u_t + cu_x = 0
\]  

where subscripts denote derivation with respect to \( t \) and \( x \) respectively. The constant \( c \) could even be scaled away, but we leave it in, mainly to allow for tracing the physical dimensions, if desirable.

Let us consider a wave that initially, i.e. at \( t = 0 \), has a sinusoidal profile: \( \sin(kx) \), \( \cos(kx) \), or a linear combination of both, which we conveniently represent in the complex notation

\[
  u(x, 0) = e^{ikx}
\]
where \( k \) is any real number, called the wave number, then
\[
u(x, t) = e^{ikx}e^{-ikct} = e^{ik(x-ct)}
\]  
(3)
so at every moment the wave is a translation of the initial shape: the wave is running to the right with velocity \( c \). Alternatively, when looking at the wave at a fixed position \( x \), we see a harmonic (sinusoidal) motion in time with one fixed frequency \( \omega \):
\[
\omega = kc.
\]  
(4)
Therefore the wave is called \textit{monochromatic}, a term from the theory of light waves, where one frequency selects one specific colour.

\textbf{Remark.} A systematic way to find solutions is to try a separated solution, i.e. a function of \( x \) only, multiplied with a function of \( t \) only: \( u(x, t) = f(x)g(t) \). The initial condition requires \( f \) to be: \( f(x) = e^{ikx} \), and from the differential equation we find that \( g \) takes the form \( e^{ikct} \). Or, as a further shortcut: once we are convinced that the time-function \( g \) is an exponential, we adopt the Ansatz
\[
u(x, t) = e^{ikx-\omega t}
\]  
(5)
and by substituting in the differential equation we find how \( \omega \) relates to \( k \).

\textbf{Dissipation}

If we add a second order derivative to the differential equation:
\[
u_t + cu_x = \nu u_{xx}
\]  
(6)
and apply the Ansatz (5), we find \(-i\omega = -ick - k^2\nu\), so the solution is
\[
u(x, t) = e^{ik(x-ct)}e^{-k^2\nu t}.
\]  
(7)
We recognize the translation wave (3), but now with an extra time factor which is a real exponential. As in most physical situations the parameter \( \nu \) is positive, the exponential is decreasing in time, so it represents damping or dissipation. It is important to note that this damping is dependent on the wave number \( k \): small wavelength ripples damp much faster than long wavelength structures. [This phenomenon is well known for the heat equation, which is in fact found from (6) by taking \( c = 0 \).]
Dispersion

If we extend the translation equation (1) with a third order derivative:

$$u_t + cu_x + \varepsilon u_{xxx} = 0 \quad (8)$$

the same procedure leads to

$$\omega = ck - \varepsilon k^3 \quad (9)$$

from which we find the solution:

$$u(x, t) = u(x, t) = e^{ik(x - (c - \varepsilon k^2)t)} \quad (10)$$

Comparing with (3), we see the solution is again a harmonic, monochromatic wave (no damping), however with a modified velocity: $c - \varepsilon k^2$. Note that this modification of velocity is depending on $k$: for positive $\varepsilon$ (as is occurring in waves), the speed decreases with wave length: small wave lengths run slower than long ones.

**Remark 1.** Relation (9) is called the dispersion relation. In fact, already (4) was a dispersion relation too. For equation (6) the dispersion relation will have imaginary terms, denoting that there is dissipation.

**Remark 2.** Another dispersive term met in differential equations is the mixed derivative $u_{xxt}$, which is typical for Boussinesq wave models [4, 5].

The equation

$$u_t + cu_x + \varepsilon u_{xxt} = 0 \quad (11)$$

has the dispersion relation

$$\omega = \frac{c - \varepsilon k^2}{a + \varepsilon k^2} \quad (12)$$

For $k$ being small which is an assumption in Boussinesq models this formula agrees with (9); but the nice property of (12) is that for larger wave number $k$ the wave velocity is still positive (and goes to zero), while from (9) velocities become negative which is unphysical.

Zero$^{th}$ order derivative

In this overview of what the effect is of spatial derivatives on wave behavior, we can even consider a term with the 0$^{th}$ derivative, that is the function itself. Such a term occurs e.g. as the friction term in the partial differential equation

$$u_t + cu_x + fu = 0,$$
where the friction factor $f$ is positive and in the simplest case a constant. Then a solution is

$$u(x,t) = e^{-ft}e^{ik(x-ct)}$$

as everyone can easily verify. The effect is exponential uniform damping, i.e. a damping that is independent of the wave number $k$.

We remember that the second derivative also gives damping of amplitude, but in a selective way: depending on the wave number. Similar difference is found between the third and first derivative, but then with respect to the wave speed.

We give a summarizing table 1

<table>
<thead>
<tr>
<th></th>
<th>lowest order derivative</th>
<th>higher order derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>even order</td>
<td>$0^{th}$: uniform damping</td>
<td>$2^{nd}$: selective damping</td>
</tr>
<tr>
<td>odd order</td>
<td>$1^{st}$: uniform speed</td>
<td>$3^{rd}$: selective speed</td>
</tr>
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**Nonlinear steepening**

Nonlinearity can, in principle, occur in any term of any equation. But if we say: the nonlinear term, we always mean a term like $uu_x$, the nonlinear advection term. Let us consider the simple nonlinear equation

$$u_t + uu_x = 0 \quad (13)$$

Compared with equation (1) where $c$ is the wave velocity, we now see that $u$ itself is the velocity. Therefore, higher values of $u$ will run faster than lower ones, wave crests run faster than troughs. As a consequence, leading slopes of a wave will get steeper, and trailing slopes will stretch [6]. The wave will turn over, and eventually break, like we can observe in the surf zone, as depicted in figure 1. (Actually, the effect is depending on water depth: the smaller the depth, the stronger the steepening.)

Let us give a mathematical support of this verbal description of the steepening phenomenon. Take a curve $x = x(t)$ in the $x,t$-plane. Then, considered along this curve, the solution $u(x,t)$ becomes a function of $t$ only: $u = u(x(t),t)$. (See figure 3, left.) The full time-derivative of this function is:

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t} = u_t + \frac{dx}{dt}u_x \quad (14)$$
Now in view of (13) we can say: if \( \frac{dx}{dt} = u \) everywhere along this curve, then (14) is 0, which implies that \( u(x(t), t) \) is a constant along the curve. But as \( u \) is the slope of the curve, this curve must be a straight line, its slope being given by the value of \( u(x(t), t) \) at \( t = 0 \), that is: the initial data at the point where the line intersects the \( x \)-axis. So from every point on the initial line, the \( x \)-axis, we can see how the value of \( u \) propagates into the \( x, t \)-plane, as illustrated in next figure 3 (right), the higher values of \( u \) running faster, the lower values running slower, and thus creating a more steeper wave front.

Combinations of terms

Dispersion and dissipation have been introduced as linear effects, acting on a mono-chromatic wave. As a general wave can be represented by a Fourier series or Fourier integral, through the principle of superposition we can imagine the result for any wave profile. This is no longer true in nonlinear equations like (13). In fact, starting with a monochromatic wave (2), the nonlinear term will become \( e^{ikx} ike^{ikx} = ike^{2ikx} \), so it generates a wave with double wave number. But as soon as \( u \) is a sum of a wave \( e^{ikx} \) and a wave \( e^{i2kx} \), the nonlinear term will generate the triple wave number and so on; so in no time all wave numbers are present in the solution. This is called the cascade effect. If, at one fixed moment, we see a wave having a steep slope, we can say that in its Fourier series all coefficients happen to be such that all terms contribute to this steepness. But if dispersion or dissipation plays a role, the coefficients will all be affected differently as time proceeds, and will soon be unable to maintain the steepness. So dispersion
and dissipation are counteracting to nonlinear steepening in this sense. The two model equations for this type of counteraction are equation
\[ u_t + uu_x = \nu u_{xx}, \quad \text{the Burger equation} \tag{15} \]
combining nonlinear steepening with dissipation, and
\[ u_t + uu_x + \varepsilon u_{xxx}, \quad \text{the Korteweg de Vries equation} \tag{16} \]
where nonlinear steepening is combined with dispersion. There is a world of literature on these equations. We will no further go into the interaction of two terms, as the aim of this chapter was to recognize the effect of single terms only.

3. SHALLOW WATER EQUATION

The section will be about the conservation properties of the Shallow Water Equations. In their simplest form, these equations apply to propagation of small amplitude disturbances over the water surface of a one-dimensional channel, i.e. a situation where quantities like elevation and velocity are assumed to be functions of (time and of) only one spatial coordinate, \( x \) say, and are therefore invariant in the other horizontal coordinate \( y \). So we may think of the flow being arbitrary wide in \( y \)-direction, but from modelling point of view it is often easy to fix the width, e.g. to unity. In that case, integration in \( y \)-direction is just a multiplication with 1, and consequently there is equivalence between a rectangle in the \( x, z \)-plane with area \( B \) and the three-dimensional block perpendicular on it, having length 1 in \( y \)-direction, for which \( B \) is the value of the volume. Therefore we may use the word volume when in the mathematics a surface would suffice, and we might use the word area when in the picture only a one-dimensional line is seen. After this explanation we will, in the sequel, no longer mention any integration over \( y \).
Continuity equation

In this one-dimensional channel consider a volume V bounded by two vertical cross sections, chosen at two arbitrary positions \( x = x_1 \) and \( x = x_2 \) respectively, as depicted in figure ?? . The total mass of water within this volume is the integral of the density over V:

\[
M = \int_{x_1}^{x_2} \int_0^{h(x,t)} \rho \, dz \, dx = \int_{x_1}^{x_2} h(x,t) \, dx \tag{17}
\]

We will make use of the conservation of mass in the volume V while the cross sections bounding V are moving along with the flow. Since the water velocity, \( u = u(x,t) \), is not dependent on \( z \), it may help our imagination if we would think of the volume V as being bounded by two vertical sheets of plastic (impermeable and light enough not to influence the momentum of the flow): they will keep vertical and keep being the bounds of the water volume V, while their positions \( x_1 \) and \( x_2 \) are moving with time: \( x_1 = x_1(t) \) and \( x_2 = x_2(t) \). The conservation of mass within V is mathematically expressed by stating that during the motion the value of \( M \) is independent of time, so in view of (17):

\[
0 = \frac{d}{dt}M = \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} h(x,t) \, dx \tag{18}
\]

Note that \( t \) is not only in the integrand \( h(x,t) \) but also in the integral bounds; so when performing the differentiation with respect to \( t \) we get (Leibniz rule that you learned in second years Calculus, where you always wondered if such a complicated expression would ever occur in practice!):

\[
0 = \int_{x_1(t)}^{x_2(t)} \frac{dh(x,t)}{dt} \, dx + h(x,t)|_{x=x_2} \frac{dx_2(t)}{dt} - h(x,t)|_{x=x_1} \frac{dx_1(t)}{dt} \tag{19}
\]

Now \( \frac{dx_1(t)}{dt} \) is the speed of the first plastic sheet, which equals the speed of the water \( u(x,t) \) at \( x = x_1(t) \); similar for \( x_2 \). So the last two terms of (19) can jointly be written as an integral again:

\[
h(x,t)|_{x=x_2} \frac{dx_2(t)}{dt} - h(x,t)|_{x=x_1} \frac{dx_1(t)}{dt} = \int_{x_1}^{x_2} \partial_x \{ h(x,t)u(x,t) \} \, dx \tag{20}
\]

Combined with the first term of (19) we have:

\[
\int_{x_1}^{x_2} [\partial_t h(x,t) + \partial_x \{ h(x,t)u(x,t) \} ] \, dx = 0 \tag{21}
\]
Remembering that $x_1$ and $x_2$ are chosen arbitrarily, we can conclude that the integrand must be zero for all $x$:

$$\partial_t h(x, t) + \partial_x \{ h(x, t) u(x, t) \} = 0 \quad (22)$$

This equation is called the continuity equation, or the mass balance, or the conservation of mass.

**Continuity equation**

For the same section $[x_1, x_2]$ of our channel we consider the total momentum contained within the volume $V$. In fact we restrict to the horizontal component. The horizontal momentum of a unit volume at position $(x, z, t)$ is $\rho u(x, t)$, so the total momentum contained within the section is the integral over $V$:

$$Mom(t) = \int_{x_1(t)}^{x_2(t)} \int_0^{h(x,t)} \rho u(x, t) \, dz \, dx = \rho \int_{x_1(t)}^{x_2(t)} h(x, t) u(x, t) \, dx \quad (23)$$

Now this total horizontal momentum will change with time due to forces acting on it. Neglecting internal and external frictions or excitations, we are left with the forces exerted by the hydrostatic pressure. These are calculated as follows. The hydrostatic pressure at a point $(x, z)$ is equal to (suppressing the notation of the time variable): $p(x, z) = \rho g (h(x) - z)$, where $h(x) - z$ is the height of the water column above the point $(x, z)$. Now pressure is the value of the force acting per unit area perpendicular on it; so at position $z$ on a plastic sheet, $\rho g (h - z)$ is the force acting in horizontal direction on the sheet per unit area, and the total force $F$ on the sheet is the integral of the pressure:

$$F = F(x) = \int_0^{h(x,t)} \rho g (h(x) - z) \, dz = \frac{1}{2} \rho g h(x,t)^2 \quad (24)$$

The force acting on the volume $V$, i.e. on the sheet $x = x_1$ by the surrounding liquid is in positive $x$-direction and is therefore taken positive: it will positively contribute to accelerating the water in $V$ in positive $x$-direction; the force on the volume $V$ at $x = x_2$ acts in negative direction; so the net
result of these two forces on $V$ is:

$$F(x_1) - F(x_2) = \frac{1}{2} \rho g h(x_1, t)^2 - \frac{1}{2} \rho g h(x_2, t)^2$$

$$= -\frac{1}{2} \rho g \{ h(x_2, t)^2 - h(x_1, t)^2 \}$$

$$= -\frac{1}{2} \rho g \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial x} \{ h(x, t)^2 \} \, dx$$

$$= -\rho g \int_{x_1(t)}^{x_2(t)} h(x, t) \frac{\partial}{\partial x} h(x, t) \, dx \quad (25)$$

Having found the net force $F$ we come to expressing the momentum balance. The net force $F$ is equal to the increase per second of the total momentum $\text{Mom}$ within the volume $V$:

$$\frac{d}{dt} \text{Mom}(t) = \rho \frac{d}{dt} \int_{x_1(t)}^{x_2(t)} h(x, t) u(x, t) \, dx$$

$$= \rho \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t} \{ h(x, t) u(x, t) \} \, dx$$

$$+ \rho h(x, t) u(x, t) \big|_{x=x_2} \frac{dx_2(t)}{dt} - \rho h(x, t) u(x, t) \big|_{x=x_1} \frac{dx_1(t)}{dt} \quad (26)$$

Now, like we already noticed after eq. (19), the factor $\frac{dx_2(t)}{dt}$ is the speed at which the position $x_2$ moves and that is by definition equal to $u(x_2(t), t)$; and similar for $x_1$; so the last two terms of (26) together give:

$$\rho h(x, t) u(x, t)^2 \big|_{x=x_2} - \rho h(x, t) u(x, t)^2 \big|_{x=x_1} = \rho \int_{x_1}^{x_2} \frac{\partial}{\partial x} \{ h(x, t) u(x, t)^2 \} \, dx \quad (27)$$

and so, when equating (25) to (26), we get

$$-\rho g \int_{x_1(t)}^{x_2(t)} h(x, t) \frac{\partial}{\partial x} h(x, t) \, dx = \rho \int_{x_1(t)}^{x_2(t)} \frac{\partial}{\partial t} \{ h(x, t) u(x, t) \} \, dx$$

$$+ \rho \int_{x_1}^{x_2} \frac{\partial}{\partial x} \{ h(x, t) u(x, t)^2 \} \, dx \quad (28)$$

Here all terms concern the same integration: $\int_{x_1(t)}^{x_2(t)}$, so when shifting all terms to one side of the equation we can combine them into one integral which should be equal to zero; remembering that $x_1$ and $x_2$ are arbitrary, we must conclude that the integrand vanishes:

$$\frac{\partial}{\partial t} \{ h(x, t) u(x, t) \} + \frac{\partial}{\partial x} \{ h(x, t) u(x, t)^2 \} + gh(x, t) \frac{\partial}{\partial x} h(x, t) = 0 \quad (29)$$
This is the momentum equation. In combination with the continuity equation (22) it can be further simplified as follows. Omitting the variables \((x, t)\), we first repeat (22) and (29):

\[
\partial_t h + \partial_x(hu) = 0 \quad (30)
\]
\[
\partial_t(hu) + \partial_x(hu^2) + gh\partial_x h = 0 \quad (31)
\]

and we expand the latter as follows:

\[
h\partial_t u + u\partial_t h + u\partial_x(hu) + hu\partial_x u + gh\partial_x h = 0 \quad (32)
\]

Now the second and third term of this equation (32) are recognized as a multiple of (30) and so they vanish; then the remaining equation can be divided through by \(h\), leading to:

\[
\partial_t u + u\partial_x h + g\partial_x h = 0 \quad (33)
\]

Here we arrive at the standard form of the (one-dimensional) Shallow Water Equations:

\[
\begin{align*}
\partial_t h + u\partial_x h + h\partial_x u &= 0 \\
\partial_t u + u\partial_x u + g\partial_x h &= 0
\end{align*} \quad (34)
\]

Note that both equations have the differential operator \(\partial_t + u\partial_x\), which is the time derivative observed when moving along with the flow. Mathematically it is a kind of directional derivative in the \(x, t\)-plane. In the next subsection however we will meet another interesting directional derivative.

4. RIEMANN INVARIANTS

Matrix-vector analysis will reveal the basic structure of the SWE (34). We will do this analysis on the linearized form, as this is already enough to show the essentials, while going into the nonlinear details might absorb too much of the readers attention. Therefore, we linearize the quantities \(h\) and \(u\):

\[
h(x, t) = H + \eta(x, t) \quad (35)
\]

where \(H\) is the constant water height of the fluid in steady flow, and \(\eta\) is the elevation measured from this equilibrium level; and:

\[
u(x, t) = a + v(x, t) \quad (36)
\]
Figure 3: The basic structure of the SWE

where $a$ is the constant flow velocity of the river (for a non-flowing channel we have $a = 0$), and $v$ is the flow velocity in excess of $a$. Like $u$, both $a$ and $v$ are supposed to be independent on the spatial coordinates $y$ and $z$.

Substituting (35) and (36) into (34) and assuming that the new dependent variables $\eta$ and $v$ are small enough to neglect second order terms (that is: products and squares of $\eta$ and $v$ and their derivatives), will lead to the linearized SWE in these new variables:

$$\begin{align*}
\partial_t \eta + a \partial_x \eta + H \partial_x v &= 0 \\
\partial_t v + a \partial_x v + g \partial_x \eta &= 0
\end{align*}$$

(37)

or, in matrix-vector notation:

$$\begin{bmatrix}
\eta \\
\v
\end{bmatrix}_t +
\begin{bmatrix}
a & H \\
g & a
\end{bmatrix}
\begin{bmatrix}
\partial_x
\end{bmatrix}
\begin{bmatrix}
\eta \\
\v
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
$$

(38)

this matrix will be named: $C$.

Eigenvalues $\lambda$ follow from

$$(a - \lambda)^2 = gH \rightarrow \lambda_{\pm} = a \pm c$$

(39)

with

$$c \equiv \sqrt{gH}$$

(40)

Eigenvector for $\lambda_+$ from:

$$\begin{bmatrix}
\partial_t \\
\tau
\end{bmatrix} 
\begin{bmatrix}
-c & H \\
g & -c
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix} \rightarrow \tau \equiv \frac{c}{g} = \frac{H}{c} = \sqrt{\frac{H}{g}}$$

(41)

and the eigenvector for $\lambda_-$ is $[1 \tau]$. 
Note from (40) and (41) that $c$ has the physical dimension of velocity and $\tau$ has the physical dimension of time.

Writing the two eigenvectors in a $2 \times 2$ matrix $T$: \[
\begin{bmatrix}
1 & \tau \\
1 & -\tau
\end{bmatrix}
\begin{bmatrix}
\eta \\
v
\end{bmatrix}
\]
this $T$ satisfies $TC = \Lambda T$, where $\Lambda$ is the diagonal matrix containing the eigenvalues. So when multiplying system (38) with this $T$ from the left we get \[
\begin{bmatrix}
1 & \tau \\
1 & -\tau
\end{bmatrix}
\begin{bmatrix}
\eta \\
v
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

or, line by line:
\[
\begin{aligned}
(\partial_t + (a + c)\partial_x)(\eta + \tau v) &= 0 \\
(\partial_t + (a - c)\partial_x)(\eta - \tau v) &= 0
\end{aligned}
\]

Now it is important to note the difference with the original system (37). In the original (linear) SWE, each of the equations contains a mix of both partial derivatives of both $\eta$ and $v$. However, in the newly derived form we do two typical observations. Let us look at (43). Firstly, it has only one differential operator, and this operator $\partial_t + (a + c)\partial_x$ is a directional derivative in the $x,t$-plane; it is the time derivative measured along a line (curve) $x = x(t)$ that has direction $\frac{dx}{dt} = a + c$. This curve is called a characteristic, and in this case where $a$ and $c$ are constants, characteristics are straight lines: $x = (a + c)t + x_0$. And secondly, this differential operator acts on one specific combination of $\eta(x,t)$ and $v(x,t)$, namely the quantity $\eta(x,t) + \tau v(x,t)$. So, considered along a characteristic, the equation (43 top) is in fact an ordinary differential equation for one dependent variable. And in this case it is very easy to see the solution: $\eta(x,t) + \tau v(x,t)$ is conserved along a right-running characteristic. Similarly, from (43 bottom) we have: $\eta(x,t) - \tau v(x,t)$ is conserved along a left-running characteristic.

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