DETERMINATION OF THE ROBUST CONDITION ON CONSENSUS SYSTEM UNDER THE COMMUNICATION CONSTRAINTS

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Abstract. This paper discusses about the robust consensus problem of a multi-agent systems under the communication constraints described by a communication graph. The uncertainty of each agent is modeled as a norm-bounded multiplicative uncertainty. A necessary and sufficient condition for achieving the robust consensus is characterized in terms of the eigenvalues of the weighted Laplacian of the communication graph. In the case where the nominal transfer function shared by all the agents is positive real, the robust consensus condition turns out to depend only on the largest eigenvalue. Since \( p(s) \) is positive real and \( \lambda_N > 0 \), the lower bound of the stability margin is given by \( \delta \geq 1 \). The stability margin \( \delta \) increases for a smaller \( \lambda_N \). Hence, when \( p(s) \) is fixed, the optimal stability margin can be obtained by minimizing \( \lambda_N \).

Keywords: Robust consensus, Multi-agent system, Laplacian matrix, Communication graph

1. INTRODUCTION

Multi-agent system came from the 'Distributed Artificial Intelligence'. The rapid advances in information technology bring good impact on the development of multi-agents that have can be easily implemented in practice. There
are so many examples of applications that implement multi-agent systems, such as ACORN (Agen-base Community Oriented Routing Network) and FIPA-OS created by the Foundation for Intelligent Physical Agents. Not only in the field of computers or networking, multi-agent systems are also widely used in other disciplines, such as physics, biology, robotics, genetics, etc.

Multi-agent system can be defined as a collection of intelligent agents (software or program) which interact and work together to achieve a goal. Durfee et al. [3] defined a multi-agent system as a network of problem solvers that work together to solve problems that can not be solved by the capacity of the individual. A multi-agent system is defined as a system consisting of a population of autonomous agents that interact with each other to achieve the same goal, and at the same time pursuing the objective of each agent individually.

One of the research is to analyze robustness in multi-agent system. As in the study Takaba [5] derived a sufficient condition for robust output synchronization against gain-bounded uncertainties for the case where the nominal agents are incrementally passive nonlinear systems. In the case of LTI multi-agent system, Hara and Tanaka [4] derived a condition for robust stability by using a generalized frequency variable representation.

Graph $G = (V, E)$ is used to represent a consensus problem in multi-agent system. In this case, the set of $V = \{v_1, v_2, \cdots, v_N\}$ represents the agents, whereas the set of $E = \{e_1, e_2, \cdots, e_N\}$ represents the relationship or communication that occur between agent $i$th and $j$th agent. Generally, matrices are often used to represent a graph. In this paper, the author use a Laplacian matrix($L$) to represent communication between each agent. Chung [2] in his book “Spectral Graph Theory” has been explained several theories about the Laplacian matrix.

In this paper, the author consider the necessary and sufficient condition for achieving robust consensus of an LTI (Linear Time Invariant) multi-agent system with norm-bounded uncertainties. We will show that the robust consensus condition can be simplified in the case where each agent is nominally passive. We will also study the stability margin for several specific graph topologies and their asymptotic properties as the number of agents goes to infinity.
2. PROBLEM FORMULATION

In the previous chapter, it has explained that the issue of consensus that will be presented related to the multi-agent system. A multi-agent system consisting of \( N \) agents can be represented as in figure 1.

![Multi agent system](image)

Each agent is described by the input-output equations:

\[
\begin{align*}
x_i &= \hat{p}_i(s)u_i, \quad y_i = x_i + v_i, \quad i = 1, 2, \ldots, N \\
\end{align*}
\]

where \( x_i, y_i, u_i \) and \( v_i \) are the controlled output, the measured output, the control input, and the measurement noise, respectively. Meanwhile, \( \hat{p}_i(s) \) is a SISO (Single-Input-Single-Output) transfer function which indicates the dynamics of each agent in the system.

\[
\hat{p}_i(s) = (1 + \Delta_i(s))p(s)
\]

where \( p(s) \) denotes the nominal dynamics shared by all agents.

The transfer function contains multiplicative uncertainty \( \Delta_i(s) \) which represents modeling errors and/or heterogeneity in the individual agents. It can be seen in figure 2.

By using (2), the system equations in (1) is reduced to:

\[
\begin{align*}
z_i &= p(s)u_i \\
x_i &= z_i + w_i \\
y_i &= x_i + v_i \\
w_i &= \Delta_i(s)z_i \\
i &= 1, 2, \ldots, N
\end{align*}
\]
Assume that $\Delta_i(s)$ is a stable transfer function whose $\mathcal{H}_\infty$-norm is bounded by a constant $\delta$ that belongs to the set $B_\delta$ defined by:

$$B_\delta := \{ \Delta \in \mathcal{H} \mid \|\Delta\|_\infty \leq \delta \}$$

In this case, $\delta$ can be interpreted as the distance between the agent $i$ with agent $j$.

Figure 2: Agent with multiplicative uncertainty

Defined some vector matrix as follows:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}$$

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

and $\Delta(s) = \text{diag}(\Delta_1(s), \Delta_2(s), \cdots, \Delta_N(s))$

Thus, (3) is equivalently rewritten as:

$$z = p(s)u$$

$$x = z + w$$

$$y = x + v$$

$$w = \Delta(s)z$$  \hspace{1cm} (4)
2.1 Communication Graph

Communication among the agents is performed through the network defined by the undirected graph $G = (V, E)$, where $V = \{1, 2, \ldots, N\}$ is the set of nodes, and $E \subseteq \{(i, j) | i, j \in V\}$ is the set of edges. The node $i \in V$ represents the $i$th agent. The edge $(i, j) \in E$ represents two-way communication link between agent $i$ and $j$. At each time, agent $i$ transmits information $y_i$ to its neighbors $j, (i, j) \in E$.

The assumption was that the communication graph $G = (V, E)$ is linear time-invariant, it means that the communication between agent $i$ and agent $j$ does not change over time.

Suppose chosen an orthonormal basis and $Q \in \mathbb{R}^{(n-1) \times n}$ is a matrix formed by that basis. $Q$ satisfies,

\[
Q 1_N = 0 \quad (5)
\]

\[
QQ^\top = I_{N-1} \quad (6)
\]

\[
Q^\top Q = \phi \quad (7)
\]

\[
\phi = I_N - \frac{1}{N} 1_N 1_N^\top \quad (8)
\]

Then,

\[
U = \begin{bmatrix} Q^\top & \frac{1}{\sqrt{N}} 1_N \end{bmatrix} \quad (9)
\]

where matrix $U$ satisfies

\[
U^\top L(G) U = \begin{bmatrix} \tilde{L} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_2 & 0 & | & 0 \\ \vdots & \ddots & | & \vdots \\ 0 & \ldots & \lambda_N & | & 0 \\ - & - & - & - & - \\ 0 & \ldots & 0 & | & 0 \end{bmatrix} \quad (10)
\]

Lemma 2.1 Suppose that $u_i, v_i \in L_2, \ i = 1, \ldots, N$. Defined that,

\[
u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad \tilde{u} = Qu, \quad \tilde{v} = Qv
\]
Thus, 
\[
\langle \tilde{u}, \tilde{v} \rangle = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle u_i - u_j, v_i - v_j \rangle
\]

### 2.2 Linear Consensus Protocol

The consensus problem is to design a protocol (distributed control law) which drives the outputs of all agents towards the same value, i.e., \( y_i - y_j, i, j \in V \) should converge to zero, or should be sufficiently small in the presence of measurement noises. Linear consensus protocol is defined as equation (11).

\[
u_i = -\sum_{j \in N_i} a_{ij} (y_i - y_j)
\]

or equivalently,

\[
dot{x} = u = -Ly = -L(x + v)
\]

Thus, multi-agent system equation can be expressed by:

\[
\begin{align*}
z &= -p(s)L(x + v) \\
x &= -p(s)L(x + v) + w \\
w &= \Delta(s)z
\end{align*}
\]

From lemma 2.1, it can be seen that \( H_2 \)-norm can be obtained by this equation.

\[
\|x\|_2 = \left( \frac{1}{2N} \sum_{i,j \in V} \|x_i - x_j\|_2^2 \right)^{1/2}
\]

**Definition 2.1** The multi-agent system in eq.(13) is said to achieve the robust consensus if the closed-loop transfer function from \( \tilde{v} \) to \( \tilde{x} \) is stable for all \( \Delta_i \in B_\delta \).
3. ROBUSTNESS ANALYSIS

Equation (13) can be transformed into a product of two matrices as follows:

\[ \begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} 0 & p(s) \\ I & p(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}, \quad \text{and} \]

\[ \begin{bmatrix} w \\ x \end{bmatrix} = \begin{bmatrix} 0 & \Delta(s)p(s) \\ I & p(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \]

Then do linear fractional transformation to obtain a transfer function \( \hat{G}(s) \) and \( \hat{H}(s) \) as follows:

\[ \hat{G}(s) = - (I_N + p(s)L)^{-1} p(s)L \]
\[ \hat{H}(s) = (I_N + p(s)L)^{-1} \]

By applying these transformations, it is obtained:

\[ \begin{bmatrix} \bar{x} \\ x' \end{bmatrix} = U^\top x, \quad \begin{bmatrix} \bar{v} \\ v' \end{bmatrix} = U^\top v \]

\[ \begin{bmatrix} z \\ \bar{x} \end{bmatrix} = \begin{bmatrix} -Q^\top G(s)Q & -Q^\top G(s) \\ H(s)Q & -G(s) \end{bmatrix} \begin{bmatrix} w \\ \bar{v} \end{bmatrix} \]

then multi-agent system equation in (13) becomes (17) and (18) as below:

\[ z = -(I_N + p(s)L)^{-1} p(s)Lw - (I_N + p(s)L)^{-1} p(s)Lv \] (17)
\[ x = (I_N + p(s)L)^{-1}w - (I_N + p(s)L)^{-1} p(s)Lv \] (18)

where,

\[ G(s) = \left[ I_{N-1} + p(s)\hat{L} \right]^{-1} p(s)\hat{L}, \] (19)
\[ H(s) = \left[ I_{N-1} + p(s)\hat{L} \right]^{-1} \] (20)

Therefore, the consensus analysis for multi-agent system of (13) is equivalently reduced to the robust stability analysis of the LFT system in eq (17) and (18). It is obvious that the consensus is achieved for the nominal case \( (\Delta(s) = 0) \) if and only if \( G(s) \) is stable.

Note that \( G(s) \) is a diagonal transfer matrix. This can be seen by the definition \( \hat{L} \) in equation (10).

\[ G(s) = \text{diag}(g_2(s), g_3(s), \ldots, g_N(s)) \] (21)
\[ g_i(s) = \frac{p(s)\lambda_i}{1 + p(s)\lambda_i}, \quad i = 1, 2, \ldots, N \] (22)
Proposition 3.1 The multi-agent system in eq. (13) achieves the nominal consensus if and only if
\[ d(s) + n(s)\lambda_i, \ i = 2, 3, \ldots, N \]
is a Hurwitz polynomial, where \((d, n)\) is a pair of coprime polynomials satisfying \(p(s) = n(s)/d(s)\).

Proposition 3.2 The multi-agent system of (13) achieves the robust consensus if and only if it achieves the nominal consensus, and
\[ \left\| \frac{p(s)\lambda_i}{1 + p(s)\lambda_i} \right\|_\infty < \delta^{-1}, \ i = 2, 3, \ldots, N \]

From this condition, we can compute the stability margin (\(\delta\)) of the multi-agent system as:
\[
\delta = \max_{i \in \{2, \ldots, N\}} \left\| \frac{p(s)\lambda_i}{1 + p(s)\lambda_i} \right\|_\infty
= \max_{i \in \{2, \ldots, N\}} \inf_{p(s) \in \mathbb{R}^+} \left\| \frac{1}{p(s)\lambda_i} \right\|
\]

3.3 Consensus for nominally passive agents

As has been assumed in the previous section, each agent in the system is nominally passive. This means that the transfer function \(p(s)\) is positive real.

From proposition 3.1, it is known that \(d(s) + n(s)\lambda\) is a Hurwirtz polynomial for each \(\lambda > 0\). Thus, the requirement to obtain a nominal consensus can be seen in proposition 3.3.

Proposition 3.3 Assume that \(p(s)\) is positive real. Thus, multi-agent system in equation (13) nominal gain consensus if and only if it satisfies:

(i.) \(d(s)\) is Hurwirtz Polynomial. That is a polynomial that each coefficient is positive and the roots is 0 or negative.

(ii.) \(d(s)\) has a root on the imaginary axis and \(\lambda_2 > 0\)
In addition to the propositions (3.3), the following lemma plays an important role for the robust consensus.

**Lemma 3.2** Assume that \( p(s) \) is positive real. Then,

\[
\left| \frac{p(s)\lambda}{1 + p(s)\lambda} \right| \geq \left| \frac{p(s)\lambda'}{1 + p(s)\lambda'} \right|, \quad \forall p(s) \in \mathbb{R}^+
\]

holds for \( \lambda > \lambda' \geq 0 \)

**Proof:** The inequality in the lemma is equivalent to

\[
(1 + p(s)\lambda')^2 \lambda^2 - (1 + p(s)\lambda)^2 \lambda'^2 \geq 0 \tag{26}
\]

By direct calculation, it is obtained:

\[
(\lambda^2 - \lambda'^2) + 2\lambda\lambda'(\lambda - \lambda')p(s)
\]

Since \( p(s) \geq 0 \) and \( \lambda > \lambda' \), it is concluded that (26) is satisfied. ■

Therefore, it is obtained the following proposition regarding the robust consensus for the positive real case.

**Proposition 3.4** Assume that \( p(s) \) is positive real. Then, the multi-agent system (13) achieves the robust consensus if and only if it achieves the nominal consensus, and

\[
\left\| \frac{p(s)\lambda_N}{1 + p(s)\lambda_N} \right\|_\infty < \delta^{-1} \tag{27}
\]

Thus, the stability margin in (25) becomes:

\[
\delta = \inf_{p(s) \in \mathbb{R}^+} \left\| 1 + \frac{1}{p(s)\lambda_N} \right\|
\]

(28)

The relationship between the topology graph used is closely related to the value of the stability margin on robust consensus. This is because the eigenvalues of each used topologies are different.

5. CONCLUSION

Based on the proposition 3.3 and 3.4 in the previous section, some conclusions can be drawn as follows:
(i) The nominal consensus is achieved as long as the graph $G$ is connected ($\lambda_2 > 0$).

(ii) A necessary and sufficient condition for achieving the robust consensus against norm-bounded uncertainties is characterized in terms of the eigenvalues of the associated Laplacian matrix of the communication graph. In the case where the nominal transfer function $p(s)$ is positive real, the consensus condition turned out to depend only on the largest eigenvalue.

(iii) Since $p(s)$ is positive real and $\lambda_N > 0$, the lower bound of the stability margin is given by $\delta \geq 1$.

(iv) It is seen in lemma (3.2) that the stability margin $\bar{\delta}$ increases for a smaller $\lambda_N$. Hence, when $p(s)$ is fixed, the optimal stability margin can be obtained by minimizing $\lambda_N$. The minimization of the eigenvalue of the weighted Laplacian was considered by Boyd [1].

References


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