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Quadratic Programming Approach in the Non-Negatif Matrix Factorization

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Abstract. Non-negative Matrix Factorization is an iteration optimization algorithm. ie to decipher one matrix into several non-negative component matrices. Non-negative Matrix Factorization (FMN) serves to obtain a picture of non-negative data. There is a problem in the Non-negative Matrix Factorization that is optimization at the constraint boundary, where in the optimization solution on the constraint boundary it is necessary to do long iteration and of course very difficult and conquers a long time. Quadratic Programing is an approach to solving linear optimization problems where the constraint is linear function and its purpose function is the square of the decision variable or multiplication of the two decision variables. This method is considered to be an effective method to overcome the optimization in the Non-negative Matrix Factorization.

Keyword: Nonnegative Matrix, Matrix Factorization, Quadratic Programming

Abstrak. *Faktorisasi Matriks Non-negatif adalah algoritma optimasi iterasi. yaitu menguraikan satu matriks menjadi beberapa matriks komponen non-negatif. Faktorisasi Matriks Non-negatif (FMN) berfungsi untuk memperoleh gambaran data non-negatif. Terdapat permasalahan pada Faktorisasi Matriks Non Negatif yaitu optimasi pada batas batasan, dimana dalam penyelesaian optimasi pada batas batasan perlu dilakukan iterasi yang panjang dan tentunya sangat sulit serta memakan waktu yang lama. Quadratic Programing merupakan suatu pendekatan penyelesaian masalah optimasi linier yang kendalanya berupa fungsi linier dan fungsi tujuannya adalah kuadrat variabel keputusan atau perkalian kedua variabel keputusan. Metode ini dinilai merupakan metode yang efektif untuk mengatasi optimasi pada Faktorisasi Matriks Non-negatif.*

Kata Kunci: *Matriks Nonnegatif, Faktorisasi Matriks, Pemrograman Kuadrat.*

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1 Introduction

Non-negative Matrix Factorization is an iteration optimization algorithm. ie to decipher one matrix into several non-negative component matrices. Given the nxm data of the integer V matrix

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data with $V_{ij} > 0$and the positive integer $r < \min(n, m)$ the non-negative matrix factorization obtains 2 (two) non-negative matrices $W \in R^{nxr}$ and $H \in R^{rxm}$ like:

$$V \approx W H$$

If each column V is refresentative on an object; non-negative matrix factorization approximates it with linear combination of r base column W. Conventional approach to obtain W and H by minimizing the difference between $V \; and \; WH$:

$$\min_{W,H} \; \mathrm{f}(W, H) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(V_{ij} - (WH)_{ij}\right)^2$$

$$subject \; to \; W_{ia} \geq 0, W_{bj} \geq 0, \forall \; i, a, b, j$$

2 Preliminaries

2.1. Non-negative matrix

The non-negative matrix is a real or integer matrix $A = [a_{ij}]$ where for each element on A is a non-negative number (equal to zero or greater than zero).

2.2 Matrix Factorization

Matrix factorization is the process of breaking or decomposition of a matrix into several matrices. In the Matrix Non-negative matrix V mxn with $v_{ij} \geq 0$, it will be decomposed into two non-negative matrices $W \in and \;\; H \in R^{mxr}$ with $r < min\,(m, n)$ such that:

$$V \approx W H \qquad (1)$$

As an explanation of Matrix Factorization by using Non Negative Matrix Factorization, Given a V_{mxn} matrix, NMF will decipher the matrix V to be:

$$V_{mxn} \approx W_{mxr} \, H_{rxn}$$

where W and H are matrices with non-negative entries.

2.3. Condition of Karush-Kuhn-Tucker

In 1951 Kuhn Tucker proposed an optimization technique that could be used to search the optimum point of a constrained function. Karush Kuhn Tucker method can be used to find the optimum solution of a function regardless of the nature of the function whether linear or non linear.

2.4. Quadratic Programing

Quadratic Programing is an approach to solving linear optimization problems where constraints are linear functions and their objective function is the square of decision variables or multiplication of two decision variables (1)

$$\min f(x) = C^T X + \frac{1}{2} X^T Q X + d$$

with constraints : $AX \leq B, X \geq 0$

When the objective function f (x) is the perfect convex for all regions it is reasonable to obtain a point which is a local and also a global minimum. So in such conditions it ensures that Q is a positive definite.

2.5. Successive Quadratic Programming

Quadratic sequential programming method (SQP) is a very powerful and popular method class for solving nonlinear programming problems, especially those with strong nonlinear boundaries (3)

3 Results and Discussion

3.1. Non-Linear Program

In the application of linear programming, the important assumption to be fulfilled is that all functions are linear. This is what then gave birth to a new concept of nonlinear programming problems. According to (1) the general form of nonlinear programming is finding the value of $x = (x_1, x_2, \dots x_n$ so that:

min / max

f (x), where f (x) is a non-linear function (2)

with constraint $g_i(x)$ for every i = 1,2, ..., m (3)

and x ≥ 0 (4)

The constraint function $g_i(x)$ can be a nonlinear function or a linear function. In addition, (**x**) and the function $g_i(x)$ are functions with n variables.

3.2 Theorem 1 Terms of KKT maximization issues (Winston, 2003)

Let f(**x**) and g_i(**x**) be a maximized pattern problem. If $x = (x_1, x_2, , .. x_n)$) is an optimal solution for (**x**) $g_i(x)$, then $x = (x_1, x_2, , .. x_n)$ must satisfy (2) and there are multipliers $\lambda_1, \lambda_2, \dots, \lambda_m$ as well as slack variable $s_1, s_2, \dots, s_n$ so that it satisfies

1. $\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} + s_j$ $\quad for\ j = 1,2, \dots, n$

2. $\lambda_i [b_i - g_i(x)] = 0$ $\quad for\ i = 1,2, \dots, m$

3. $\left(\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \right) x_j$ $\quad for\ j = 1,2 \dots, n$

4. $\lambda_i \geq 0$ $\quad for\ i = 1,2,\dots,m$

5. $s_j \geq 0$ $for\ j = 1,2,\ldots,n$

Theorem 2.4. Terms of KKT minimization issues (2)

Let f(**x**) and $g_i(\mathbf{x})$ be a problem of patterned drinking. If $x = (x_1, x_2, , .. x_n)$ is an optimal solution for f(**x**) and g_i(x), then $x = (x_1, x_2, , .. x_n)$) must satisfy (2) and there are multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ as well as the surplus variable $s_1, s_2, \ldots, s_n$ so that it satisfies.

1. $\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} + s_j$ $for\ j = 1,2, \ldots, n$
2. $\lambda_i[b_i - g_i(x)] = 0$ $for\ i = 1,2, \ldots, m$
3. $\left(\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}\right) x_j$ $for\ j = 1,2 \ldots, n$
4. $\lambda_i \geq 0$ $for\ i = 1,2,\ldots,m$
5. $e_j \geq 0$ $for\ j = 1,2,\ldots,n$

On the second condition of Theorem 2 and Theorem 3 result $g_i(x) - b_i \leq 0$ *This can be seen when* $\lambda_i = 0$, *so* $[b_i - g_i(x)] \neq 0$ *Based on the general form of the constraint function, then* $[b_i - g_i(x)] > 0$ *is* $g_i(x) \leq b_i$

3.3.1. Quadratic Programming Solution

Nature.1. Complementary slackness in quadratic programming (2)

1) e_j and s_j under Kuhn-Tucker and x_j can not both be positive.

2) The surplus (excess) or slack variables for the i-th constraints and λ_i can not both be positive

Evidence of Nature.1.

1. Consider the terms 1) and 3) on the theorem 1 namely:

   condition 1) ie : $\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_1 \frac{\partial g_i}{\partial x_j} + s_j = 0$, so :

   $\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} = -s_j$ substituted to condition 3)

   $\left(\frac{\partial f}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j}\right) x_j = 0$

   $s_j x_j = 0$

If $s_j = 0$, then $x_j \neq 0$, _ ie $x_j > 0$

$x_j = 0$, then $x_j \neq 0$, ie $s_j > 0$ or $s_j < 0$. Under condition 4) then $s_j > 0$.

This applies also to Theorem 2.4, so it is evident that e_j and s_j under Kuhn-Tucker and x_j can not be both positive.

2. Look at Terms 2) ie $\lambda_i[\, b_i - g_i(x)] = 0$

In the constraint function $g_i(x) \leq b_i$ then the canonical form of the constraint is $g_i(x) + s_1' = b_i$, so that Condition 2 becomes:

$\lambda_i s_1' = 0$

If $\lambda_i = 0$ then $s_1' \neq 0$, ie $s_1' > 0$.

If $s_i' = 0$ then $\lambda_i \neq 0$, ie $\lambda_i > 0$ or $\lambda_i < 0$. Under Condition 5) then,

$\lambda_i > 0$.

In the constraint function $g_i(x) \geq b_i$ can be converted to $g_i(x) - e_i' = b_i$.

In the same way the $\lambda_i e_i' = 0$, so it is proved that the surplus (excess) or slack variables for the i-th constraints and λ_i can not both be positive.

The equations derived from the step are a step in the linearity of a nonlinear programming problem by using Kuhn Tucker's condition.

3.2 Successive Quadratic Programming

Successive Quadratic Programming (SQP) is a very powerful and popular class of methods for solving nonlinear programming problems, especially those with strong nonlinear boundaries (3). Like sequential linear programming, quadratic programming problems are formed from nonlinear programming problems and solved iteratively until they are optimized. However, iterative procedures are different from successive linear programs.

In quadratic programming, the economic model of quadratic functions, and constraints are all linear equations. To overcome this problem the Lagrangian function is formed, and Kuhn-Tucker's condition is applied to the Lagrangian function to obtain a set of linear equations at this point, it is important to understand the solution of the quadratic programming problem.

This is part of the motivation for using quadratic programming which can be demonstrated by the following equation:

$$\max : \sum_{j=1}^{n} c_j X_j - \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} q_{jk} X_j X_k \qquad (5)$$

$subject\ to$:

$$\sum_{j=1}^{n} a_{ij} X_j \leq b_i \qquad for\ i = 1,2, \dots, m$$

$$X_j \geq 0 \qquad for\ j = 1,2 \dots, n$$

where $q_{jk} = q_{kj}$ is the second partial derivative with respect to x_j and x_k model nonlinear economy.

The quadratic programming procedure starts by adding the slack variable x_{n+i} to the linear constraint equation. No need to use ${x_{n+i}}^2$ because the problem will be solved with linear programming, and all variables must be positive or zero. The Lagrangian function is formed as follows:

$$L(X,\lambda) = \sum_{j=1}^{n} c_j X_j - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk} X_j X_k - \sum_{i=1}^{m} \lambda_1 \left(\sum_{j=1}^{n} a_{ij} X_j + x_{n+1} - b_1 \right)$$

A positive Lagrange multiplier is required, so a negative sign is used in the equation with the constraint equation.

Setting the first partial derivative of a Lagrangian function with respect to x_j and i equal to zero gives the following linear linear algebraic (n + m) set:

$$c_j - \sum_{k=1}^{n} q_{jk} X_k - \sum_{i=1}^{m} a_{ij} \lambda_1 \leq 0 \qquad for\ j = 1,2, \dots n \qquad (6)$$

$$\sum_{m}^{n} a_{ij} X_j + X_{n+1} - b_1 = 0 \qquad for\ i = 1,2, \dots m \qquad (7)$$

Kuhn-Tucker's inequality form, equation 6, is used to calculate $x_j > 0$. Also, the condition of complementary clearance must be satisfied, ie the slack variable variable X_{n+1} and the Lagrange i multiplier is zero.

$$\lambda_i X_{n+1} = 0 \qquad for\ i = 1,2, \dots, m \qquad (8)$$

If X_{n+1}= 0, then the constraint is active, the equation; $\lambda_i \neq 0$. However, if $X_{n+1} \neq 0$, then the constraint is inactive, an inequality and $\lambda_i = 0$ The equations of (6) and (7) can be converted to linear programming problems in the following way. The surplus variable is added to equation (6) as sj, and the slack variable has been added to equation (7) as X_{n+i}. The slack variable X_{n+i} can serve as a variable for a base that was originally feasible for equation (7). However, artificial variables are required to have a reasonable basis for the equation (6). Adding artificial variables z_j with the coefficients c_j to equation (6) is an easy way to start with z_j.=1 Also, the objective function is to minimize the number of artificial variables to ensure that they are not in the final optimal solution As a result of this modification, equation (6) and (7) to be:

$$min : \sum_{j=1}^{n} z_j \quad (9)$$

$subject\ to$:

$$\sum_{k=1}^{n} q_{jk}\, X_k + \sum_{i=1}^{m} a_{ij}\, \lambda_i - s_j + c_j z_j = c_i \qquad for\ j = 1,2,\dots,n$$

$$\sum_{j=1}^{n} a_{ij}\, X_j + X_{n+i} = b_i \qquad for\ i = 1,2,\dots,m$$

This is now a linear programming problem that can be solved for optimal values of x and λ, the solution of quadratic programming problems. In addition, the solution must satisfy $x \geq 0, \lambda \geq 0\ and\ \lambda_i X_{n+i} = 0$

4 Conclusions

Optimization on Non Negative Matrix Factorization is generally non-linear and for solving non-linear program optimization problems can be used Quadratic Programming approach.

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