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The L(2,1) Labeling of Windmill, Binomial Tree, Pan, Gear, Prism, Helm, Lilly, Hurdle, and Key Graph

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ABSTRACT

Let G be a (s,t) graph. Let whole number-valued function $f: V(G) \to \mathrm{N}_0$ such that, a and b are two adjacent vertices in $V(G)$, then if $d(a,b) = 1, |f(a) - f(b)| \geq 2$ and if $d(a,b) = 2, |f(a) - f(b)| \geq 1$. The function f is called $L(2,1)$ labeling. The $L(2,1)$ labeling number of G, called $\lambda_{2,1}(G)$ is the smallest number k for label of G. In this research, we will further discuss the $L(2,1)$ labeling of windmill, binomial tree, pan, gear, prism, helm, lilly, hurdle, and key graph.

Keyword: L(2,1) Labeling, Windmill, Binomial Tree, Pan, Gear

1. Introduction

Graphs defined here are finite, undirected, simple, and connected. The notion of $L(2,1)$ labelling is described by Griggs and Yeh (1992). Griggs and Yeh define the $L(2,1)$ labelling of the graph G as a function f that maps every $a,b \in V(G)$ a label of the set of whole number such that $|f(a) - f(b)| \geq 2$ if $d(a,b) = 1$ and $|f(a) - f(b)| \geq 1$ if $d(a,b) = 2$ []. Moreover, they examine $\lambda_{2,1}$ on graphs of cycles, trees, stars. Some other recent results of $L(2,1)$ labelling researches were done by Bantva about $L(2,1)$ labelling in the context of some graph operations [1], Calamoneri about $L(2,1)$ labelling of unigraphs [2], Komarullah about $L(2,1)$ labelling of Vertex Amalgamation [5], Kusbudiono, Umam, Halikin, and Fatekurohman about $L(2,1)$ labelling of Lollipop and Pendulum graph [6], Lum about upper bounds on the $L(2,1)$ labelling number of graphs with maximum degree Δ [7], Mitra and Bhoumik about $L(2,1)$ labelling of circulant graphs [8], Panda and Goel about $L(2,1)$ labelling of perfect elimination bipartite graphs [9], Sagala about $L(2,1)$ labelling of Sierpinski graph [13], Shao and Vesel about $L(2,1)$ labelling of the strong product of paths and cycles [14], Shao and Solis-Oba about labelling total graphs with a condition at distance two [15], Shao and Yeh about $L(2,1)$ labelling and operations of graphs [16], Shao, Yeh, and Zhang about $L(2,1)$ labelling on the graphs and the frequency assignment problem [17], Zhang, Yang, and Li H. about $L(2,1)$ labelling of the brick product graphs

[18]. In this research, we will further discuss the $L(2,1)$ labeling and determine the minimum span value of windmill, binomial tree, pan, gear, prism, helm, lilly, hurdle, and key graph.

2. Research Methods

This research uses a literature study research method. The steps are as follows: first, discuss the research problem; read the literature sources that contain $L(2,1)$ labelling; analyze the open problem, then make a pattern and theorem about it including the proof: draw the graph started with the small amount of vertices until we obtain the pattern so that we can determine the upper and lower bounds of $L(2,1)$ labelling of graphs as theorems, then we prove the theorem from looking at the subgraph, and formulate the results to be the conclusion.

We use the following known results.

Proposition 1 [15] Let P_n be a path with $n \geq 2$ vertices. Then $\lambda_{2,1}(P_2) = 2, \lambda_{2,1}(P_3) = \lambda_{2,1} = P_4 = 3$, and $\lambda_{2,1}(P_n) = 4$, for $n \geq 5$.

Proposition 2 [15] Let C_n be a cycle with $n \geq 3$ vertices. Then $\lambda_{2,1}(C_n) = 4$.

Lemma 1 [17] If H is a subgraph of G, then $\lambda_{2,1}(H) \leq \lambda_{2,1}(G)$.

Proposition 3 [15] Let K_n be a complete graph with n vertices. Then $\lambda_{2,1}(K_n) = 2n - 2$.

Proposition 4 [15] Let $K_{1,n}$ be a star graph with $n + 1$ vertices. Then $\lambda_{2,1}(K_{1,n}) = n + 1$.

3. Results and Discussion

3.1 L(2,1) Labeling of Windmill Graph

Definition 1 The windmill graph W_n^m is the graph obtained by taking m copies of the K_n with a vertex in common. [4]

Theorem 1 Let $m, n \geq 3$ be positive integers. If W_n^m be the windmill graph with $m, n \geq 3$, then

$$\lambda_{2,1}(W_n^m) = mn - m + 1.$$

Proof. Let $V(W_n^m) = \{v_0\} \cup \{v_1^1, v_1^2, \dots, v_1^m\} \cup \dots \cup \{v_{n-1}^1, v_{n-1}^2, \dots, v_{n-1}^m\}$ and $E(W_n^m) = \{v_0 v_i^j | 1 \leq i \leq n-1,\ j \in [1, m]\} \cup \{v_i^j v_i^k | 1 \leq i \leq n-1, k \in [1, m]\}$. Since W_n^m contains $K_{1,mn-m}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(W_n^m) \geq \lambda_{2,1}(K_{1,mn-m}) = mn - m + 1$. Next, to prove $\lambda_{2,1}(W_n^m) \leq mn - m + 1$, we give the label 0 to the central vertex and give the label 2, 3, 4, …, $m + 1$ to the first vertex on every subgraph K_n, respectively. Continuing give the labels $m + 2, m + 3, \dots, 2m + 1$ to the second vertex on every subgraph K_n, respectively. Doing this process until all the vertices had been labelled. So that we have the greatest label $m(n - 1) + 1$ since the windmill graph have m- copies of K_n and a vertex in common (can be written by m copies of K_{n-1} where all vertices of the copies K_n are connecting with the central vertex) and the label 2, 3, 4, 5, …, respectively for the vertices of subgraphs K_{n-1} and 0 for the central vertex, then we must adding the $\lambda_{2,1}(W_n^m)$ with 1. Thus $\lambda_{2,1}(W_n^m) \leq mn - m + 1$. ∎

3.2 L(2,1) Labeling of Binomial Tree Graph

Definition 2 The binomial tree graph B_m is the graph obtained from two binomial trees B_{m-1} that are linked together: the root of one is the leftmost child of the root of the other. [12]

Theorem 1 Let $m \geq 2$ be positive integers. If B_m be the binomial tree with $m \geq 2$, then

$$\lambda_{2,1}(B_m) = m + 1.$$

Proof. Let $V(B_m) = \{u_0, u_1, u_2, \dots, u_m\} \cup \{u_1^1, u_2^1, \dots, u_{(m-1)!}^1\} \cup \{u_1^2, u_2^2, \dots, u_{(m-2)!}^2\} \cup \dots \cup \{u_{(m-(m-1))!}^m\}$ and the pattern for the edges are illustrated in figure 1 . Since B_m contains $K_{1,m}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(B_m) \geq m+1$. Next, to prove $\lambda_{2,1}(B_m) \leq m+1$, we give the label 0 for the central vertex and 2, 3, 4, …, m respectively to the vertices that adjacent with the central vertex. Give the label to the vertex that wanna be labelled from the smaller number of vertices that adjacent with. For labelling this graph, we just playing the trick with the vertices that adjacent with the vertex that we want to label it. Labelling starting from the top rank vertices to the lowest rank vertices respectively. If the label of any vertex is 0 so the labels of vertices that adjacent to it are 2, 3, 4, …, respectively, give the label to the vertex that not yet been labelled from the smaller number of vertices that adjacent with. Next, if the label of any vertex is 1 so the labels of vertices that adjacent to it are 3, 4, 5, …, respectively. If the label of any vertex is 2 so the labels of vertices that adjacent to it are 4, 5, 6, …, respectively. Do this things until all vertices had been labelled. Then we will see that $u^1_{(m-1)!} = m+1$ is the vertex with the greatest label and the labels of other vertices are smaller than it.■

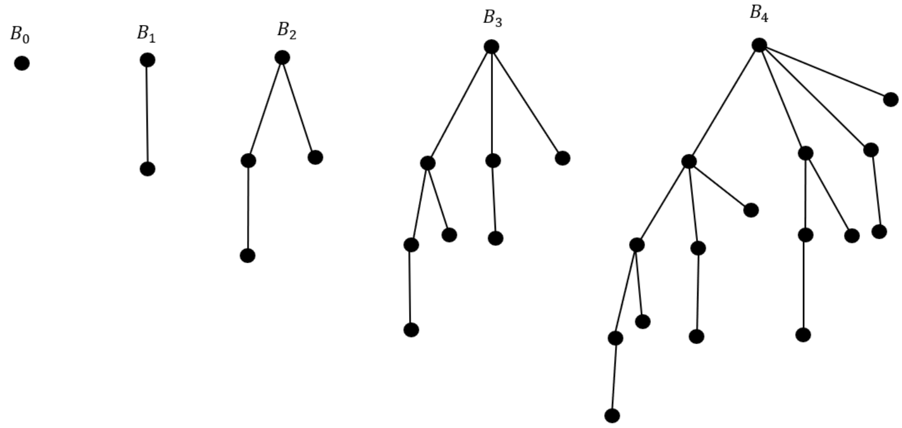

Figure 1. Binomial Tree B_0, B_1, B_2, B_3, B_4.

3.3 L(2,1) Labeling of Pan Graph

Definition 3 The pan graph Pa_n is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.

Theorem 3 Let $n \geq 3$ be positive integers. If Pa_n be the pan graph with $n \geq 3$, then

$$\lambda_{2,1}(Pa_n) = 4.$$

Proof. Let $V(Pa_n) = \{u_0, u_1, u_2, \dots, u_n\}$ and $E(Pa_n) = \{u_0u_1\} \cup \{u_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{u_nu_1\}$. Since Pa_n contains C_n, then by Proposition 2 and Lemma 1, $\lambda_{2,1}(Pa_n) \geq \lambda_{2,1}(C_n) = 4$. Next, to prove $\lambda_{2,1}(C_n) \leq 4$, defined a labelling function f of Pa_n as follow.

$$f(u_0) = \begin{cases} 2, & for\ n \equiv 1\ mod\ 3; \\ 3, & for\ n \equiv 0\ mod\ 3; \\ 4, & for\ n \equiv 2\ mod\ 3. \end{cases}$$

$$For\ n \equiv 1\ mod\ 3, f(u_1) = 0, f(u_2) = 3, f(u_3) = 1, f(u_4) = 4,$$

$$f(u_i) = \begin{cases} 0, & for\ i \equiv 2\ mod\ 3; \\ 2, & for\ i \equiv 0\ mod\ 3; \\ 4, & for\ i \equiv 1\ mod\ 3. \end{cases}$$

$$For\ n \equiv 0\ mod\ 3,$$

$$f(u_i) = \begin{cases} 0, & for\ i \equiv 1\ mod\ 3; \\ 2, & for\ i \equiv 2\ mod\ 3; \\ 4, & for\ i \equiv 0\ mod\ 3. \end{cases}$$

$$For\ n \equiv 2\ mod\ 3, f(u_1) = 3, f(u_2) = 1, f(u_3) = 4, f(u_4) = 2, f(u_5) = 0,$$

$$f(u_i) = \begin{cases} 4, & for\ i \equiv 0\ mod\ 3; \\ 2, & for\ i \equiv 1\ mod\ 3; \\ 0, & for\ i \equiv 2\ mod\ 3. \end{cases}$$

So f is $L(2,1)$ 80abelling of Pa_n for $n \geq 3$. Therefore, $\lambda_{2,1}(Pa_n) \leq 4$. ■

3.4 L(2,1) Labeling of Gear Graph

Definition 4 The gear graph G_n is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. [4]

Theorem 4 Let $n \geq 3$ be positive integers. If G_n be the gear graph with $n \geq 3$, then

$$\lambda_{2,1}(G_n) = \begin{cases} 6, & for\ n = 3, 4, 5; \\ n+1, & for\ n \geq 6. \end{cases}$$

Proof. Let $V(G_n) = \{u_0\} \cup \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ and $E(G_n) = \{u_0u_i | 1 \leq i \leq n\} \cup \{u_iv_i | 1 \leq i \leq n\} \cup \{v_iu_{i+1} | 1 \leq i \leq n\} \cup \{v_nu_1\}$. Let's start with the first case, since G_3 contains $K_{1,3}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(G_3) \geq \lambda_{2,1}(K_{1,3}) = 4$. So, we have 0, 1, 2, 3, 4 for labelling the vertices of G_3. Trying all the possibility labelling, the labels for $u_0, u_1, u_2, u_3, v_1, v_2, v_3$ are 0, 2, 3, 4, v_1, 1, v_3; since there will be two vertices that have no label by Pigeonhole Principle, so we try adding 5 as the label and with the same way, trying all the possibility labelling, we have 0, 2, 3, 4, 5, 1, v_3; 0, 2, 3, 5, 5, 1, v_3; 0, 2, 4, 5, v_1, 1, v_3; 0, 3, 4, 5, 1, 2, v_3; 1, 3, 4, 5, 0, 2, v_3, respectively. By the label 0 until 5, we still have at least a vertex that have no label by Pigeonhole Principle. So, $\lambda_{2,1}(G_3) \geq 6$. Next, to prove $\lambda_{2,1}(G_3) \leq 6$ can be seen from the figure 2. Next, since G_4 contains $K_{1,4}$, then by Proposition 4 and Lemma 1 $\lambda_{2,1}(G_4) \geq \lambda_{2,1}(K_{1,4}) = 5$. So, we have 0, 1, 2, 3, 4, 5, for labelling the vertices of G_4. Next, trying all the possibility labelling, the labels for $u_0, u_1, u_2, \dots, u_4$ and $v_1, v_2, \dots, v_4$ are 0, 2, 3, 4, 5, v_1, v_2, 1, v_4; 0, 4, 2, 5, 3, v_1, v_2, v_3, 1; 5, 0, 1, 2, 3, v_1, v_2, v_3, v_4; 5, 0, 2, 1, 3, 4, v_2, v_3, v_4, respectively, so there will be at least three vertices that have no label by Pigeonhole Principle, then $\lambda_{2,1} \geq 5$. For $n = 5$, since G_5 contains $K_{1,5}$ then by Proposition 4 and Lemma 1, $\lambda_{2,1}(G_5) \geq 6$. Next, to prove $\lambda_{2,1}(G_n) \leq 6$ for $n = 4, 5$, we can see from the figure 2. Therefore, $\lambda_{2,1}(G_n) \leq 6$ for $n = 4, 5$. For the second case, since G_n contains $K_{1,n}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(G_n) \geq \lambda_{2,1}(K_{1,,n}) = n + 1$. Next, to prove $\lambda_{2,1}(G_n) \leq n + 1$, defined a labelling function of G_n as follow.

$$f(u_0) = 0, f(u_i) = i + 1, \quad i \in [1, n], f(v_1) = 5, f(v_i) = \begin{cases} \quad 1, & for\ i \equiv 0\ mod\ 2; \\ 2, & for\ i \equiv 1\ mod\ 3, \quad i \geq 3 \end{cases}$$

$$f(v_n) = 4.$$

So f is $L(2,1)$ labelling of G_n for $n \geq 6$. Therefore, $\lambda_{2,1}(G_n) \leq n + 1$. ■

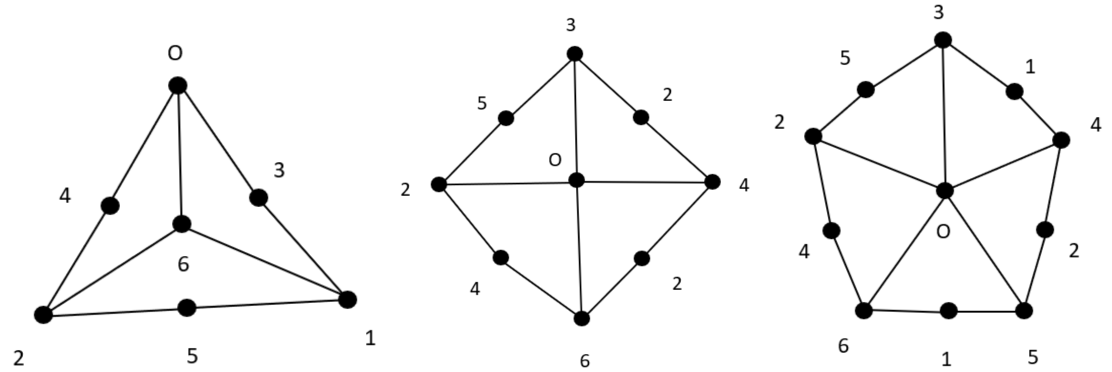

Figure 2. $L(2,1)$ Labeling of G_3, G_4, G_5.

3.5 L(2,1) Labeling of Prism Graph

Definition 5 The prism graph Pr_n is a graph corresponding to the skeleton of an n-prism.

Theorem 5 Let $n \geq 3$ be positive integers. If Pr_n be the prism graph with $n \geq 3$, then

$$\lambda_{2,1}(Pr_n) = \begin{cases} 5, & for\ n = 3; \\ 7, & for\ n \geq 4. \end{cases}$$

Proof. Let $V(Pr_n) = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ and $E(Pr_n) = \{u_iu_{i+1} | 1 \leq i \leq n-1\} \cup \{u_nu_1\} \cup \{v_iv_{i+1} | 1 \leq i \leq n-1\} \cup \{v_nv_1\}$. Let's start with the first case, since Pr_3 contains $K_{1,3}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(Pr_3) \geq \lambda_{2,1}(K_{1,3}) = 4$. Trying all the possibility labelling with the pigeonhole principle with the labels 0, 1, 2, 3, 4, then there will be at least a vertex that has no labelling, so $\lambda_{2,1}(Pr_3) \geq 5$. Next, to prove $\lambda_{2,1}(Pr_3) \leq 5$, we can see from the figure 3. Next for the second case, since Pr_n contains $K_{1,3}$, then by Proposition 4 and Lemma1, $\lambda_{2,1}(Pr_n) \geq \lambda_{2,1}(K_{1,3}) = 4$. So we 0, 1, 2, 3, 4, by trying all the possibility labelling with the Pigeonhole Principle to the label $u_1, u_2, \ldots, u_n$. For $n \equiv 1 \bmod 3$, we get the pattern of the labels such as 0, 3, 1, 4, 0, 2, 4, …, 0, 2, 4 for the internal vertices, then we will get at least a vertex that has no label by Pigeonhole Principle. For $n \equiv 0 \bmod 3$, we get the pattern of the labels such as 0, 2, 4, …, 0, 2, 4 for the internal vertices, then we will get at least a vertex that has no label by pigeonhole principle. For $n \equiv 2 \bmod 3$, we get the pattern of the labels such as 1, 3, 0, 2, 4, …, 0, 2, 4 for the internal vertices, then we will get at least a vertex that has no label by Pigeonhole Principle. If we try adding 5 and 6 to the possibility labelling, with the same way, we will get at least a vertex that has no label by Pigeonhole Principle. This applies to $n \equiv 0, 1, 2 \bmod 3$, so $\lambda_{2,1}(Pr_n) \geq 7$. Next, to prove $\lambda_{2,1}(Pr_n) \leq 7$, defined a labelling function f of Pr_n as follow.

$$For\ n \equiv 1 \bmod 3, f(u_1) = 0, f(u_2) = 3, f(u_3) = 1, f(u_4) = 4,$$

$$f(u_i) = \begin{cases} 0, & for\ i \equiv 2 \bmod 3; \\ 2, & for\ i \equiv 0 \bmod 3; \\ 4, & for\ i \equiv 1 \bmod 3. \end{cases}$$

$$f(v_i) = \begin{cases} 2, & for\ i \equiv 1 \bmod 3; \\ 5, & for\ i \equiv 2 \bmod 3;\ f(v_{n-1}) = 3 \\ 7, & for\ i \equiv 0 \bmod 3. \end{cases}$$

$$For\ n \equiv 0 \bmod 3, f(u_i) = \begin{cases} 0, & for\ i \equiv 1 \bmod 3; \\ 2, & for\ i \equiv 2 \bmod 3; \\ 4, & for\ i \equiv 0 \bmod 3. \end{cases} f(v_i) = \begin{cases} 3, & for\ i \equiv 1 \bmod 3; \\ 6, & for\ i \equiv 2 \bmod 3; \\ 1, & for\ i \equiv 0 \bmod 3. \end{cases}$$

$$For\ n \equiv 2 \bmod 3, f(u_1) = 3, f(u_2) = 1, f(u_3) = 4, f(u_4) = 2, f(u_5) = 0,$$

$$f(u_i) = \begin{cases} 4, & for\ i \equiv 0 \bmod 3; \\ 2, & for\ i \equiv 1 \bmod 3; f(v_1) = 7, f(v_2) = 5, f(v_3) = 0, f(v_4) = 6, f(v_n) = 4, \\ 0, & for\ i \equiv 2 \bmod 3. \end{cases}$$

$$f(v_i) = \begin{cases} 3, & for\ i \equiv 2 \bmod 3; \\ 1, & for\ i \equiv 0 \bmod 3; \\ 6, & for\ i \equiv 1 \bmod 3. \end{cases}$$

So f is $L(2,1)$ labelling of Pr_n for $n \geq 4$. Therefore, $\lambda_{2,1}(Pr_n) \leq 7$. ■

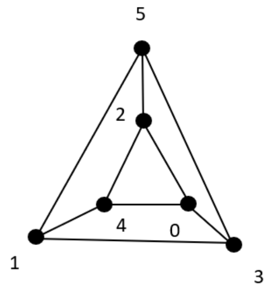

Figure 3. $L(2,1)$ labelling of Pr_3.

3.6 L(2,1) Labeling of Helm Graph

Definition 6 The helm graph H_n is the graph obtained from an n-wheel graph by adjoining a pendant edge at each node of the cycle.

Theorem 6 Let $n \geq 3$ be positive integers. If H_n be the helm graph with $n \geq 3$, then

$$\lambda_{2,1}(H_n) = \begin{cases} 6, & for\ n = 3, 4; \\ n+1, & for\ n \geq 5. \end{cases}$$

Proof. Let $V(H_n) = \{u_0\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\}$ and $E(H_n) = \{u_0 u_i | 1 \leq i \leq n\} \cup \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{u_i v_i | 1 \leq i \leq n\}$. Started from the first case, since H_n contains $K_{1,4}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(H_n) \geq \lambda_{2,1}(K_{1,4}) = 5$. Next, for the second case, since H_n contains $K_{1,n}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(H_n) \geq \lambda_{2,1}(K_{1,n}) = n+1$. Next, to prove $\lambda_{2,1}(H_n) \leq n+1$, for $n = 3, 4$ can be easily seen by figure 4 and for $n \geq 5$, we give the label 0 to the central vertex, then give the label 2, 3, 4, …, $n+1$,respectively, to the first vertex then skip the second vertex and continue label the third vertex with 3 then skip the next vertex and continue label the next to the next vertex with 4, so the trick for labelling $u_1, u_2, \dots, u_n$ is skipping the next vertex and give the label to the next from the next vertex, this was done since the difference label of two adjacent vertices are at least 2. Last the labels for $v_1, v_2, \dots, v_n$ are 5 for v_1 and the others are 1 (case H_3), 4 for v_1 and the others are 1 (case H_4), 6 for v_1 and the others are 1 (case H_5), 4 for v_1 and the others are 1 (case H_n for $n \geq 6$). Therefore, $\lambda_{2,1}(H_n) = n + 1$. ■

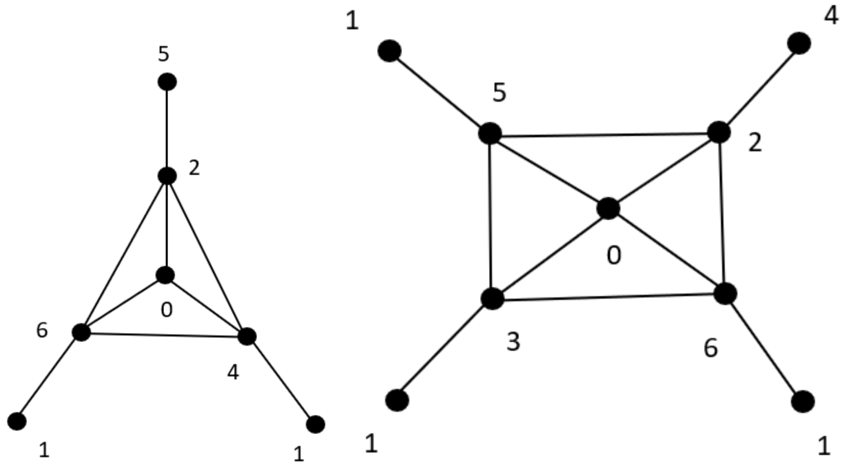

Figure 4. $L(2,1)$ labelling of H_3, H_4.

3.7 L(2,1) Labeling of Lilly Graph

Definition 7 The lilly graph L_n $(n \geq 2)$ is the graph obtained by joining two star graphs and two paths P_n $(n \geq 2)$. [10]

Theorem 7 Let $n \geq 2$ be positive integers. If L_n be the lilly graph with $n \geq 2$, then

$$\lambda_{2,1}(L_n) = 2n + 3.$$

Proof. Let $V(L_n) = \{a\} \cup \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\} \cup \{w_1, w_2, \dots, w_n\} \cup \{x_1, x_2, \dots, x_n\}$ and $E(L_n) = \{au_i | 1 \leq i \leq n\} \cup \{av_i | 1 \leq i \leq n\} \cup \{aw_1\} \cup \{w_i w_{i+1} | 1 \leq i \leq n-1\} \cup \{ax_1\} \cup \{x_i x_{i+1} | 1 \leq i \leq n-1\}$. Since L_n contains $K_{1,2n+2}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(L_n) \geq \lambda_{2,1}(K_{1,2n+2}) = 2n + 3$. Next, to prove $\lambda_{2,1}(L_n) \leq 2n + 3$, defined a labelling function f of L_n as follow.

$$f(a) = 0, f(v_i) = i + 2, \quad i \in [1, n], \quad f(u_i) = i + 4, \quad i \in [1, n],$$

$$f(w_i) = \begin{cases} 0, & for\ i \in [1, n], \quad i \equiv 1\ mod\ 5; \\ 6, & for\ i \in [1, n], \quad i \equiv 2\ mod\ 5; \\ 4, & for\ i \in [1, n], \quad i \equiv 3\ mod\ 5; \\ 1, & for\ i \in [1, n], \quad i \equiv 4\ mod\ 5; \\ 3, & for\ i \in [1, n], \quad i \equiv 0\ mod\ 5. \end{cases}$$

$$f(x_i) = \begin{cases} 0, & for\ i \in [1, n], \quad i \equiv 1\ mod\ 5; \\ 2, & for\ i \in [1, n], \quad i \equiv 2\ mod\ 5; \\ 4, & for\ i \in [1, n], \quad i \equiv 3\ mod\ 5; \\ 1, & for\ i \in [1, n], \quad i \equiv 4\ mod\ 5; \\ 3, & for\ i \in [1, n], \quad i \equiv 0\ mod\ 5. \end{cases}$$

So f is $L(2,1)$ labelling of L_n for $n \geq 2$. Therefore, $\lambda_{2,1}(L_n) \leq 2n + 3$. ■

3.8 L(2,1) Labeling of Hurdle Graph

Definition 8 The hurdle graph Hd_n with $n - 2$ hurdles is a graph obtained from a path P_n by attaching pendant edges to every internal vertices of the path. [11]

Theorem 8 Let $n \geq 3$ be positive integers. If Hd_n be the hurdle graph with $n \geq 3$, then

$$\lambda_{2,1}(Hd_n) = \begin{cases} 4, & for\ n = 3, 4; \\ 5, & for\ n \geq 5. \end{cases}$$

Proof. Let $V(Hd_n) = \{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_{n-2}\}$ and $E(Hd_n) = \{u_i u_{i+1} | 1 \leq i \leq n-1\} \cup \{u_{i+1} v_i | 1 \leq i \leq n-2\}$. Started with the first case, $Hd_3 \cong K_{1,3}$ so it is clear. For $n = 4$, since Hd_4

contains $K_{1,3}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(Hd_4) \geq \lambda_{2,1}(K_{1,3}) = 4$. Next, to prove $\lambda_{2,1}(Hd_4) \leq 4$, defined a labelling function f of Hd_n as follow.

$$f(u_1) = f(u_4) = 2, f(u_2) = 0, f(u_3) = 4, f(v_1) = 3, f(v_2) = 1.$$

So f is $L(2,1)$ labelling of Hd_4. Therefore, $\lambda_{2,1}(Hd_4) \leq 4$. Next for the second case, since Hd_n contains $K_{1,3}$, then by Proposition 4 and Lemma 1, $\lambda_{2,1}(Hd_n) \geq \lambda_{2,1}(K_{1,3}) = 4$. So we have the labels 0, 1, 2, 3, 4. Trying all the possibility label for the graph, then we will have the pattern:

$$\begin{bmatrix} 0 & 2\,v_2 & 3\,v_4 & 0 \\ 3 & 0\;4 & 1\,u_5 & u_6 \end{bmatrix}, \begin{bmatrix} 0 & 4\,v_2 & v_3\,0 & 0 \\ 2 & 0\;3 & 1\;4 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 4\,v_2 & 1\;2 & 0 \\ 3 & 0\;2 & 4\;0 & 3 \end{bmatrix}$$

Then we will get at least a vertex that has no label that suffices the $L(2,1)$ condition. So, $\lambda_{2,1}(Hd_n) \geq 5$. Next, to prove $\lambda_{2,1}(Hd_n) \leq 5$, defined a labelling function f of Hd_n as follows.

$$f(u_1) = 2, f(u_i) = \begin{cases} 0, & for\ i \equiv 2 \bmod 3; \\ 5, & for\ i \equiv 0 \bmod 3; \\ 3, & for\ i \equiv 1 \bmod 3. \end{cases} f(v_i) = \begin{cases} 3, & for\ i \equiv 1 \bmod 4; \\ 1, & for\ i \equiv 2,3 \bmod 4; \\ 2, & for\ i \equiv 0 \bmod 4. \end{cases}$$

So f is $L(2,1)$ labelling of Hd_n for $n \geq 5$. Therefore, $\lambda_{2,1}(Hd_n) \leq 5$. ■

3.9 L(2,1) Labeling of Key Graph

Definition 9 The key graph ky_n is the graph obtained from K_2 by appending one vertex of C_5 to an endpoint and Hoffman tree $P_n \odot K_1$ to the other endpoint of K_2. [11]

Theorem 9 Let $n \geq 2$ be positive integers. If ky_n be the key graph with $n \geq 2$, then

$$\lambda_{2,1}(ky_n) = 4.$$

Proof. Let $V(ky_n) = \{u_1, u_2, \dots, u_5\} \cup \{v_1, v_2, \dots, v_n\} \cup \{w_1, w_2, \dots, w_n\}$ and $E(ky_n) = \{u_i u_{i+1} | 1 \leq i \leq 4\} \cup \{u_5 u_1\} \cup \{u_1 v_1\} \cup \{v_i v_{i+1} | 1 \leq i \leq n-1\} \cup \{v_i w_i | 1 \leq i \leq n\}$. Since ky_n contains C_5, then by Proposition 2 and Lemma 1, $\lambda_{2,1}(ky_n) \geq \lambda_{2,1}(C_5) = 4$. Next, to prove $\lambda_{2,1}(ky_n) \leq 4$, defined a labelling function f of ky_n as follows.

$$f(u_1) = 0, f(u_2) = 2, f(u_3) = 4, f(u_4) = 1, f(u_5) = 3,$$

$$f(v_i) = \begin{cases} 4, & for\ i \equiv 1 \bmod 4; \\ 2, & for\ i \equiv 2 \bmod 4; \\ 3, & for\ i \equiv 3 \bmod 4; \\ 0, & for\ i \equiv 0 \bmod 4. \end{cases} f(w_i) = \begin{cases} 1, & for\ i \equiv 1,3 \bmod 4; \\ 0, & for\ i \equiv 2 \bmod 4; \\ 2, & for\ i \equiv 0 \bmod 4. \end{cases}$$

So f is $L(2,1)$ labelling of ky_n for $n \geq 2$. Therefore, $\lambda_{2,1}(ky_n) \leq 4$. ■

3. Conclusion

In this paper, we determined the smallest span value of the $L(2,1)$ labelling of windmill, binomial tree, pan, gear, prism, helm, lilly, hurdle, and key graph.

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