

On Certain Type of Sequence Spaces Defined by φ -Function

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Abstract. In this paper, we introduce non-negative real valued φ -function on \mathbb{R} . Using φ -function, we define the sequence spaces $W(f)$, $W_0(f)$, and $W_\infty(f)$. We will study some topological properties defined by certain paranorm of these spaces.

Keyword: Paranorm, Sequence Space

Abstrak. Pada makalah ini, kami memperkenalkan fungsi- φ bernilai real non-negatif pada \mathbb{R} . Menggunakan fungsi- φ , kami mendefinisikan ruang barisan $W(f)$, $W_0(f)$, dan $W_\infty(f)$. Kami mempelajari beberapa sifat topologi didefinisikan atas paranorma tertentu pada ruang ini.

Kata Kunci: Paranorm, Ruang Barisan

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1. Introduction

The space of all sequence take value on real numbers we denote by ω . Any non-empty linear subspace of ω is called a sequence space. The sequence spaces l_1 , cs and bs , we use for the meaning of the spaces of all absolutely convergent series, convergent series, and bounded series, respectively.

A linear topological space X over the real field \mathbb{R} is said to be a *paranormed space* if there is a function $g : X \rightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, and scalar multiplication is continuous, that is $|\lambda_n - \lambda| \rightarrow 0$ and $g(x_n - x) \rightarrow 0$ imply $g(\lambda_n x_n - \lambda x) \rightarrow 0$ for every λ in \mathbb{R} and x in X , where θ is the zero in the linear space X .

A paranorm g is called *total paranorm* if $g(x) = 0$ implies $x = 0$ and the pair $\mathbb{X} = (X, g)$ is called *total paranormed space*. Wilansky [1, p. 183] showed that by given some total paranorm, any set become a linear space or vector space.

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Nakano [2] and Simons [3] introduced the notion of paranormed sequence space. Later on it was further investigated by some author, like Maddox [4,5], Lascarides [6], Rath and Tripathy [7], Tripathy and Sen [8] and many others ([9], [10], [11]).

A function $M: \mathbb{R} \rightarrow [0, \infty)$ is said to be an *Orlicz function* if M is even, convex, continuous, $M(0) = 0$, and $M(u) \rightarrow \infty$ as $u \rightarrow \infty$.

W. Orlicz [12] used the idea of Orlicz function to construct the space L^M . Lindenstrauss-Tzafriri [13] construct the sequences space $\ell^{\exists}(M)$ make use of the Orlicz function M ;

$$\ell^{\exists}(M) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) \left(\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right) \right\}$$

The set denoted by $\ell^{\exists}(M)$ is called an Orlicz sequence space. Lindenstrauss-Tzafriri proved that $\ell^{\exists}(M)$ is a Banach space respected with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

The Orlicz sequence space $\ell^{\exists}(M)$ with $M(x) = x^p$ is closely link to the space ℓ_p for $1 \leq p < \infty$,

$$\ell_p = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k|^p < \infty \right\}.$$

Various modifications of these definitions have been studied in the mathematical literature, like ℓ is changed to another solid sequence space. If any sequence (x_k) in a sequence space X and for all sequence (λ_k) of scalar with $|\lambda_k| \leq 1$ for all natural numbers k , implies $(\lambda_k x_k) \in X$, then the sequence space X is said to be *solid* (or normal) [14].

The Δ_2 – *condition* be valid for an Orlicz function M , if there exists positive real number K such that for every positive real number x implies $M(2x) \leq KM(x)$.

A continuous function $f: \mathbb{R} \rightarrow [0, \infty)$ is called a φ – *function* if $f(t) = 0$ if and only if $t = 0$, even and non-decreasing on $[0, \infty)$. Using φ – function, we define the following sets

$$W(f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) (\exists l > 0) \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k - l|}{\rho}\right) \rightarrow 0, n \rightarrow \infty \right\}$$

$$W_0(f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k|}{\rho}\right) \rightarrow 0, n \rightarrow \infty \right\}$$

$$W_{\infty}(f) = \left\{ x = (x_k) \in \omega : (\exists \rho > 0) \sup_n \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

In this work we will study some of topological properties of the set $W(f)$, $W_0(f)$ and $W_{\infty}(f)$.

2. Main Results

In this section we prove some results involving the set $W(f)$, $W_0(f)$ and $W_{\infty}(f)$.

Theorem 1. *The set $W(f)$, $W_0(f)$ and $W_{\infty}(f)$ are linear space, if f as φ -function fills the Δ_2 -condition.*

Proof. Let $x = (x_k)$ and $y = (y_k)$ be sequences in $W(f)$ so there exists $\rho_1, \rho_2 > 0$ and $l_1, l_2 > 0$ with the result that

$$\frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k - l_1|}{\rho_1}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad (1)$$

and

$$\frac{1}{n} \sum_{k=1}^n f\left(\frac{|y_k - l_2|}{\rho_2}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty \quad (2)$$

Let $\rho = \max\{\rho_1, \rho_2\}$ and assume that $l = l_1 + l_2$. since f is a non-decreasing function on $[0, \infty)$, then

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k + y_k - (l_1 + l_2)|}{\rho}\right) &\leq \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k - l_1|}{\rho}\right) + \frac{1}{n} \sum_{k=1}^n f\left(\frac{|y_k - l_2|}{\rho}\right) \\ &\leq \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k - l_1|}{\rho_1}\right) + \frac{1}{n} \sum_{k=1}^n f\left(\frac{|y_k - l_2|}{\rho_2}\right). \end{aligned}$$

From (1) and (2), we get

$$\frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k + y_k - (l_1 + l_2)|}{\rho}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Thus, the addition of $x + y$ closed in $W(f)$. (3)

Let $\alpha \in \mathbb{R}$ and sequence x in $W(f)$, then we can possess $\rho > 0$ and $l > 0$ so as

$$\frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k - l|}{\rho}\right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Let $l = \alpha l_1$. For $\alpha = 0$, it can be easily verified that

$$\frac{1}{n} \sum_{k=1}^n f\left(\frac{|\alpha x_k - l|}{\rho}\right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Then we assume that $\alpha \neq 0$. Since $0 < |\alpha|$, then by the Archimedian, there exists $n_0 \in \mathbb{N}$ so that $|\alpha| \leq 2^{n_0}$, and because of f is a non-decreasing function on $[0, \infty)$ and satisfy the Δ_2 -condition, then there exists $M > 0$ such that $f(|\alpha|x_k) \leq f(2^{n_0}x_k) \leq M^{n_0}f(x_k)$, for any natural numbers k . Thus,

$$\begin{aligned} \frac{1}{n} \sum_{k=1}^n f\left(\frac{|\alpha x_k - l|}{\rho}\right) &= \frac{1}{n} \sum_{k=1}^n f\left(\frac{|\alpha||x_k - l_1|}{\rho}\right) \\ &\leq \frac{M^{n_0}}{n} \sum_{k=1}^n f\left(\frac{|x_k - l_1|}{\rho}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (4)$$

From (3) and (4), we can take the conclusion that the set $W(f)$ is a linear space.

The proof of the rest cases, $W_0(f)$ and $W_\infty(f)$ will follow similarly. ■

Theorem 2. A real function $g : W(f) \rightarrow \mathbb{R}$ becomes a paranorm if we define $g(x)$ as

$$g(x) = \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k|}{\rho} \right) \leq 1, n \in \mathbb{N} \right\}.$$

Proof. It is not hard to show that $g(x) \geq 0$ and $g(-x) = g(x)$, for every $x \in W(f)$. Let a sequence $x = (x_k), y = (y_k) \in W(f)$, then there is positive real numbers ρ_1, ρ_2 and $l_1, l_2 > 0$ with the result

$$\frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k - l_1|}{\rho_1} \right) \rightarrow 0, \text{ as } n \rightarrow \infty$$

and

$$\frac{1}{n} \sum_{k=1}^n f \left(\frac{|y_k - l_2|}{\rho_2} \right) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Since f is a non-decreasing function on $[0, \infty)$, we get

$$\begin{aligned} g(x+y) &= \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k + y_k|}{\rho} \right) \leq 1 \right\} \\ &\leq \inf \left\{ \rho_1 > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k|}{\rho_1} \right) \leq 1 \right\} + \inf \left\{ \rho_2 > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|y_k|}{\rho_2} \right) \leq 1 \right\} \\ &\leq g(x) + g(y). \end{aligned}$$

So the following inequality $g(x+y) \leq g(x) + g(y)$ holds, for every $x, y \in W(f)$. Furthermore, for any scalar sequence (λ_n) and $(x_k^{(n)}) \subset W(f)$, where

$$|\lambda_n - \lambda| \rightarrow 0 \text{ and } g(x_k^{(n)} - x_k) \rightarrow 0 \text{ for } x \in W(f) \text{ and } n \rightarrow \infty$$

we have

$$\begin{aligned} g(\lambda_n x_k^{(n)} - \lambda x_k) &= \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|\lambda_n x_k^{(n)} - \lambda x_k|}{\rho} \right) \leq 1 \right\} \\ &\leq \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|\lambda_n x_k^{(n)} - \lambda x_k^{(n)}|}{\rho} \right) \leq 1 \right\} \\ &\quad + \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|\lambda x_k^{(n)} - \lambda x_k|}{\rho} \right) \leq 1 \right\} \\ &= \inf \left\{ \rho = \left(\frac{|\lambda_n - \lambda|}{|\lambda_n - \lambda|} \rho \right) > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k^{(n)}|}{\rho / |\lambda_n - \lambda|} \right) \leq 1 \right\} \end{aligned}$$

$$\begin{aligned}
& + \inf \left\{ \rho = \left(\frac{|\lambda|}{|\lambda|} \rho \right) > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k^{(n)} - x_k|}{\rho/|\lambda|} \right) \leq 1 \right\} \\
& = |\lambda_n - \lambda| \inf \left\{ \rho^* = \left(\frac{\rho}{|\lambda_n - \lambda|} \right) > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k^{(n)}|}{\rho^*} \right) \leq 1 \right\} \\
& + |\lambda| \inf \left\{ \rho^{**} = \left(\frac{\rho}{|\lambda|} \right) > 0 : \frac{1}{n} \sum_{k=1}^n f \left(\frac{|x_k^{(n)} - x_k|}{\rho^{**}} \right) \leq 1 \right\} \\
& = |\lambda_n - \lambda| g(x_k^{(n)}) + |\lambda| g(x_k^{(n)} - x_k).
\end{aligned}$$

Since $|\lambda_n - \lambda| \rightarrow 0$ and $g(x_k^{(n)} - x_k) \rightarrow 0$, it follows that $g(\lambda_n x_k^{(n)} - \lambda x_k) \rightarrow 0$. This is a complete proof of the theorem. ■

Theorem 3. *The linear space $W(f)$ is a complete paranormed sequence space, whenever f as φ -function satisfies the convex property and Δ_2 -condition.*

Proof. Let an Cauchy real sequence $(x^{(n)})$ in $W(f)$ with

$$(x^{(n)}) = (x_k^{(n)}) = (x_1^{(n)}, x_2^{(n)}, \dots).$$

It's mean for any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ so that for every $m \geq n \geq n_0$, we get

$$g(x^{(m)} - x^{(n)}) < \varepsilon.$$

Thus,

$$\frac{1}{r} \sum_{k=1}^r f \left(\frac{|x_k^{(m)} - x_k^{(n)}|}{\varepsilon} \right) \leq 1.$$

Since f is convex, then

$$\frac{1}{r} \sum_{k=1}^r f(|x_k^{(m)} - x_k^{(n)}|) \leq \varepsilon \frac{1}{r} \sum_{k=1}^r f \left(\frac{|x_k^{(m)} - x_k^{(n)}|}{\varepsilon} \right) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f(|x_k^{(m)} - x_k^{(n)}|) = 0$ for every $m \geq n \geq n_0$. This implies that $|x_k^{(m)} - x_k^{(n)}| < \varepsilon$ for every $m \geq n \geq n_0$. It follows that $(x_k^{(n)})$ becomes a Cauchy sequence on \mathbb{R} for every $k \in \mathbb{N}$. Since $\mathbb{R} = (\mathbb{R}, |\cdot|)$ is a complete normed space, then there exists $x_k \in \mathbb{R}$ for every $k \in \mathbb{N}$ with $\lim_{n \rightarrow \infty} x_k^{(n)} = x_k$. Thus for every $n \geq n_0$, we get

$$|x_k^{(m)} - x_k| = |x_k^{(m)} - \lim_{n \rightarrow \infty} x_k^{(n)}| = \lim_{n \rightarrow \infty} |x_k^{(m)} - x_k^{(n)}| < \varepsilon^2.$$

Let $x = (x_k) \in \omega$. Since $(x^{(n)}) \subset W(f)$, then there exists $l > 0$ and $\rho > 0$ implies

$$\frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - l|}{\rho}\right) \rightarrow 0, \quad r \rightarrow \infty.$$

Using the continuity of f

$$\begin{aligned} \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k - l|}{\rho}\right) &= \frac{1}{r} \sum_{k=1}^r f\left(\frac{|\lim_{n \rightarrow \infty} x_k^{(n)} - l|}{\rho}\right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - l|}{\rho}\right) = 0, \quad r \rightarrow \infty. \end{aligned}$$

This implies that

$$\frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k - l|}{\rho}\right) \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

As a result, the sequence x in $W(f)$. Furthermore, we will show that $g(x^{(n)} - x) \rightarrow 0$ as $n \rightarrow \infty$. Because of the continuous property of φ -function, then

$$\begin{aligned} \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - x_k|}{\rho}\right) &= \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - \lim_{m \rightarrow \infty} x_k^{(m)}|}{\rho}\right) \\ &= \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - x_k^{(m)}|}{\rho}\right) \leq 1. \end{aligned}$$

Thus,

$$g(x^{(n)} - x) = \inf \left\{ \rho > 0 : \frac{1}{r} \sum_{k=1}^r f\left(\frac{|x_k^{(n)} - x_k|}{\rho}\right) \leq 1 \right\}.$$

This implies that $g(x^{(n)} - x) < \rho$ for every $\rho > 0$. It follows that there exists a real sequence $\left(\frac{c}{2^m}\right)$, $m \geq 1$, for a real number c together with

$$g(x^{(n)} - x) < \frac{c}{2^m}, \quad m \geq 1.$$

Thus we get $g(x^{(n)} - x) \rightarrow 0$ as $n \rightarrow \infty$. We can deduce that the linear space $W(f)$ satisfies complete property with paranorm. ■

Furthermore, in the similar way, we can conclude that the spaces $W_0(f)$ and $W_\infty(f)$ are complete paranormed spaces equipped with the same paranorm, i.e.,

$$g(x) = \inf \left\{ \rho > 0 : \frac{1}{n} \sum_{k=1}^n f\left(\frac{|x_k|}{\rho}\right) \leq 1, n \in \mathbb{N} \right\}.$$

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