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Vertex Exponent of Asymmetric Two-coloured Cycle

A. Syahmarani[1]* and S. Suwilo[1]

[1]Department of Mathematics, Universitas Sumatera Utara, Medan, 20155, Indonesia

Abstract. This paper is about an asymmetric two-coloured cycle. Let D be an asymmetric two-coloured cycle on n vertices, where n is odd and $n \geq 3$, we show that the exponent of the kth vertex of D is exactly $(n^2-1)/4+\lfloor k/2 \rfloor$.

Keywords: Exponent Digraphs, Primitive, Two-coloured Digraphs, Vertex Exponent.

Abstrak. *Paper ini membahas tentang cycle dwi-warna asimetrik. Misalkan D merupakan suatu cycle dwi-warna asimetrik dengan n vertex, dimana n ganjil dan $n \geq 3$, diperlihatkan bahwa eksponen vertex ke k dari D adalah tepat $(n^2-1)/4+\lfloor k/2 \rfloor$.*

Kata Kunci: *Eksponen Digraph, Primitif, Digraph Dwi-warna, Eksponen Vertex.*

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1 Digraphs and Two-coloured Digraphs

We discuss about vertex exponent of primitive asymmetric two-coloured digraphs with a special type. The notations and terminologies for digraphs and two-coloured digraphs in this paper is based on [1]. A *walk* of length m from vertex u to vertex v in a graph is a sequence of m of the form

$$(v_0,v_1),(v_1,v_2),\ldots,(v_{m-1},v_m)$$

where $v_0 = u$ and $v_m = v$. We state a walk w from u to v as a (u,v)-walk or w_{uv} and $\ell(w_{uv})$ is denoted by its length. A (u,v)-walk is *closed* provided $u = v$ and if not is *open*. A *path* from u to v is a walk with no repeated vertices except possibly $u = v$. A *cycle* is a closed path. A *loop* is a closed cycle of length 1.

A digraph D called *strongly connected* if for each pair of vertices u and v there is a (u,v)-walk and also a (v,u)-walk of length exactly k in D. The *exponent* of D, denoted by exp(D), is the smallest of such positive integer k. A strongly connected digraph D is primitive if and only if the greatest common divisor of lengths of all cycle in D is 1 [1]. A *symmetric* digraph D is a digraph such that the arc (u,v) is in D whenever the arc (v,u) is in D. Since a symmetric digraph must have a cycle of length 2, a symmetric digraph is primitive if and only if it contains a cycle with odd length.

*Corresponding author at: Department of Mathematics, Universitas Sumatera Utara, Medan, 20155, Indonesia

E-mail address: aghni.syahmarani@usu.ac.id

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A *two-coloured digraph* or a 2-*digraph*, is a digraph in which each of its arcs is coloured by either red or blue (2 colours). In a two-coloured digraph we differentiate a walk by how many red and how many blue arcs it contains. We intend an $(h,k)^T$-walk from u to v as a (u,v)-walk that consists of h red arcs and k blue arcs and the length is $h+k$. The vector $(r(w),b(w))^T$ is called the composition of the walk w. A two-coloured digraph D is *strongly connected* provided that its underlying digraph is strongly connected. Underlying digraph means the digraph obtained from D by ignoring its arc colour. An *asymmetric two-coloured digraph* is a symmetric two-coloured digraph for which an arc (u,v) is coloured by red whenever the arc (v,u) is coloured by blue and vice versa. A strongly connected two-coloured digraph is *primitive* provided there are nonnegative integers h and k such that for each pair of vertices u and v there is an $(h,k)^T$-walk from u to v. The smallest nonnegative integer $h+k$ among all such nonnegative integers h and k is called the 2-*exponent* of D, denoted by $\exp_2(D)$.

Let a two-coloured digraph D and $\{\delta=\delta_1,\delta_2,\ldots,\delta_t\}$ as its set of all cycle. A *cycle matrix* M of D is a 2 by t matrix whose ith column is the composition of the cycle δ_i, $i=1,2,\ldots,t$. That is

$$M=\begin{bmatrix} r(\delta_1) & r(\delta_2) & \ldots & r(\delta_t) \\ b(\delta_1) & b(\delta_2) & \ldots & b(\delta_t) \end{bmatrix}.$$

The content of M is defined to be 0 if the rank$(M)=1$ and the greatest common divisor of the 2 by 2 minors of M, otherwise.

Shader and Suwilo [1] initiated the research on 2-exponents of two-coloured digraphs. They indicated that the largest 2-exponent of primitive two-coloured digraphs on n vertices lies on the interval $[(n^3-5n^2)/3,(3n^3+2n^2-2n)/2]$. Since that time, many papers have been published on the subject. Suwilo [2] has shown the exponent of an asymmetric primitive two-coloured (n,s)-lollipop. Let D be an asymmetric primitive two-coloured (n,s)-lollipop. Since D has a red path of length $(s+1)/2+(n-s)$, then $\exp_2(D)=(s^2-1)/2+(s+1)(n-s)$. Suwilo has also shown that if n is odd and $s=n$ or $s=n-2$, then $\exp_2(D)=(n^2-1)/2$ and if n is even and $s=n-1$, then $\exp_2(D)=n^2/2$. Gao and Shao [3] have shown the generalized exponent of primitive two-coloured Wielandt digraph $W_n^{(2)}$. They showed 3 types of Wielandt digraph and found the formula to find vertex exponent for each type. Fomichev and Avezova [4] found the exact formula for the exponents of the mixing digraphs of register transformations.

Suwilo [5] also found the vertex exponent of two-colored primitive extremal ministrong digraphs $D^{(2)}$ on n vertices. If $D^{(2)}$ has one blue arc, he showed that the exponents of vertices of $D^{(2)}$ lie on $[n^2-5n+8,n^2-3n+1]$. Since $D^{(2)}$ has two blue arcs, he also showed that the exponents of vertices in $D^{(2)}$ lie on $[n^2-4n+4,n^2-n]$. Vertex exponents of a class of two-colored of the Hamiltonian digraphs have been shown in [6]. They found that the vertex exponents of primitive two-colored digraph $L_n^{(2)}$ on $n\geq 5$ vertices whose underlying digraph is the Hamiltonian digraph consisting of the cycle $v_1\rightarrow v_n\rightarrow v_{n-1}\rightarrow\cdots\rightarrow v_2\rightarrow v_1$ and the arc $v_1\rightarrow v_{n-2}$ is known that $2n^2-6n+2\geq\exp(L_n^{(2)})\geq(n^3-2n^2+1)/2$. If the $\exp(L_n^{(2)})=(n^3-2n^2+1)/2$, then its vertex exponents lie on $[(n^3-2n^2-3n+4)/4,(n^3-2n^2+3n+6)/4]$ and if $\exp(L_n^{(2)})=2n^2-6n+2$, then its vertex exponents lie on $[n^2-4n+5,n^2-2n-1]$.

The aim of this paper is to show the vertex exponent of an asymmetric primitive two-coloured cycle. We give the formula to find the vertex exponent of an asymmetric primitive two-coloured cycle as the result.

2 Vertex Exponent of An Asymmetric Two-coloured Cycle

Let D be an asymmetric primitive two-coloured cylce of length n. If D is primitive, then n have to be odd and $n \geq 3$. Furthermore, if D is asymmetric, then D has directed cycles of length 2 with composition $(1,1)^T$, and also has two directed cycles δ_1 and δ_2 of length n. The cycle are

$$(1,2),(2,4),\ldots,(n-1,n),(n,n-2),\ldots,(5,3),(3,1)$$

and

$$(1,3),(3,5),\ldots,(n,n-1),(n-1,n-3),\ldots,(4,2),(2,1).$$

Hence, the compositions form of the directed cycles of D are $(1,1)^T$, $(n-a,a)^T$, and $(a,n-a)^T$ for some nonnegative integer $0 \leq a \leq n$. This implies the cycle matrix of D is of the form

$$M = \begin{bmatrix} n-a & a & 1 & 1 & \ldots & 1 \\ a & n-a & 1 & 1 & \ldots & 1 \end{bmatrix}.$$

Since D is primitive, by [1] the content of M is 1. Therefore

$$1 = gcd(n(n-2a), n-2a, 2a-n) = \pm(n-2a).$$

This implies that $a = (n+1)/2$ or $a = (n-1)/2$, so without loss the generality we may assume that

$$M = \begin{bmatrix} (n+1)/2 & (n-1)/2 & 1 & 1 & \ldots & 1 \\ (n-1)/2 & (n+1)/2 & 1 & 1 & \ldots & 1 \end{bmatrix}.$$

We note that since D is asymmetric, every vertex of D lies on a $(1,1)^T$-walk. This means each $(r,b)^T$-walk in D can be extended to a $(r+t,b+t)^T$-walk for each positive integer $t \geq 1$.

Let D be an asymmetric primitive two-coloured cycle where the arcs colouring in succession are $(n+1)/2$ red arcs and $(n-1)/2$ blue arcs or vice versa. Since n is odd, there are two cases. For $n = 4m+1$, colour γ_1 by giving red on (v_j, v_{j-2})-arcs where $3 \leq j \leq (n+1)/2$, (v_1, v_2)-arc, and (v_i, v_{i+2})-arcs where $2 \leq i \leq (n-1)/2$ and the others with blue. For $n = 4m+3$, colour γ_1 by giving red on (v_j, v_{j-2})-arcs where $3 \leq j \leq (n+3)/2$, (v_1, v_2)-arc, and (v_i, v_{i+2})-arcs where $2 \leq i \leq (n+1)/2$ and the others with blue.

Lemma 2.1. *Suppose D be an asymmetric primitive two-coloured cycle of length n, then the exponent* v_k *of D* $exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.

Proof. Suppose D be an asymmetric primitive two-coloured cylce of length n. Since D is primitive, then n have to be odd and $n \geq 3$. Moreover, since D is asymmetric, for each v_k on D there is a $(1,1)^T$-closed walk from v_k to itself of length 2. Let $p_{k,x}$ be a path which connect v_k to v_x on D is a $(r_{p_{k,x}}, b_{p_{k,x}})^T$-path consist $r_{p_{k,x}}$ red arcs and $b_{p_{k,x}}$ blue arcs. For each v_k on D, then $(r,b)^T$-walk from v_k to v_x is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} r_{p_{k,x}} \\ b_{p_{k,x}} \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers x_1, x_2 and x_3.

Case 1. For $n = 4m+1$, we show that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$. If k is even, for each v_x on D, there is a $(r,b)^T$-walk from v_k to v_x. We take a $(r,b)^T$-walk from v_k to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k}{2}$.
There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3.
There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3. We note that if $1 \leq k < \frac{n+1}{2}$, then $e = \frac{n-1}{4} - \frac{k}{2}$, $f = 0$, $g = 0$ and $h = \frac{n-1}{4} + \frac{k}{2}$. But if $\frac{n+1}{2} \leq k \leq n$, then $e = g = \frac{k}{2} - \frac{n-1}{4}$, $f = 0$, and $h = \frac{n-1}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.1}$$

Substraction the second by the first component of Equation 2.1, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k}{2}$.

If k is odd, for each v_x on D, there is a $(r,b)^T$-walk from v_k to v_x. We take a $(r,b)^T$-walk from v_k to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k-1}{2}$.
There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3.
There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3. We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{n-1}{4} + \frac{k-1}{2}$, $f = 0$, $g = 0$ and $h = \frac{n-1}{4} - \frac{k-1}{2}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{n-1}{2}$ $f = \frac{k-1}{2} - \frac{n-1}{4}$, $g = 0$, and $h = \frac{k-1}{2} - \frac{n-1}{4}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.2}$$

Substraction the second by the first component of Equation 2.2, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k-1}{2}$. Then for each k where $1 \leq k \leq n$ we have $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.

Case 2. For $n = 4m+3$, we also show that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$. If k is even, for each v_x on D, there is a $(r,b)^T$-walk from v_k to v_x. We take a $(r,b)^T$-walk from v_k to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n+1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3.

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+3}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3. We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{n-1}{2}$, $f = \frac{n+1}{4} + \frac{k}{2}$, $g = \frac{n+1}{4} - \frac{k}{2}$ and $h = \frac{n-1}{2}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{k}{2} - \frac{n-1}{4}$, $f = 0$, $g = 0$ and $h = \frac{3n-1}{4} - \frac{k}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.3}$$

Substraction the second by the first component of Equation 2.3, then we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k}{2}$.

If k is odd, for each v_x on D, there is a $(r,b)^T$-walk from v_k to v_x. We take a $(r,b)^T$-walk from v_k to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$, then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k-1}{2}$.

There are red path of length e and blue path of length f from v_k to $v_{\frac{n+1}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3.

There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+3}{2}}$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3. We note that if $1 \leq k \leq \frac{n+1}{2}$, then $e = \frac{3n-1}{4} + \frac{k-1}{2}$, $f = 0$, $g = 0$ and $h = \frac{k-1}{2} - \frac{n+1}{4}$. But if $\frac{n+1}{2} < k \leq n$, then $e = \frac{k-1}{2} - \frac{n+1}{4}$, $f = 0$, $g = 0$, and

$h = \frac{k-1}{2} - \frac{3n-1}{4}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.4}$$

Substraction the second by the first component of Equation 2.4, then if $1 \leq k \leq \frac{n+1}{2}$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. And if $\frac{n+1}{2} < k \leq n$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \frac{k-1}{2}$. Then for each k where $1 \leq k \leq n$ we have $\exp_D(v_k) \geq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.

Lemma 2.2. *Let D be an asymmetric primitive two-coloured cycle of length n, then the exponent v_k of D $exp_D(v_k) \leq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let D be an asymmetric primitive two-coloured cylce of length n. If D is primitive, then n have to be odd and $n \geq 3$. Moreover, if D is asymmetric, for each v_k on D there is a $(1,1)^T$-closed walk from v_k to itself of length 2. For $k = 1$, there is a $(r,b)^T$-walk from v_1 to v_j with composition $(\frac{n^2-1}{8}, \frac{n^2-1}{8})^T$. We show that $\exp_D(v_1) = \frac{1}{4}(n^2-1)$. First, we show $\exp_D(v_1) \leq \frac{1}{4}(n^2-1)$. Let $P_{1,j}$ be a path which start from v_1 to v_j. There is a $(r,b)^T$-walk from v_1 to v_j. In this case, the walk that starts at v_1, moves to v_j along the path $P_{1,j}$, and moves $(r(p_{1,j}) - b(p_{1,j}))$ times around the cycle γ_1 is the shortest walk from v_1 to v_j. This composition of this walk is

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} (r(p_{1,j}) \\ b(p_{1,j}) \end{bmatrix} + (r(p_{1,j}) - b(p_{1,j})) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix}.$$

Notice that $P_{1,j}$ is a path from v_1 to v_j where $b(p_{1,j}) = 0$. Then the composition of its walk becomes

$$\begin{bmatrix} r \\ b \end{bmatrix} = (r(p_{1,j}) - b(p_{1,j})) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n+1}{2} \end{bmatrix}.$$

We find that $P_{1,j} \leq \frac{n-1}{4}$. This implies $r(p_{1,j}) + b(p_{1,j}) \leq \frac{n-1}{4}$, hence $r(p_{1,j}) - b(p_{1,j}) \leq r(p_{1,j}) + b(p_{1,j}) \leq \frac{n-1}{4}$, then

$$\begin{bmatrix} r \\ b \end{bmatrix} \leq \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Now using $(1,1)^T$-walks, we can extend the walk into $(t,t)^T$-walk with $t = \frac{n^2-1}{8}$ then

$$\exp_D(v_1) \leq \frac{1}{4}(n^2-1).$$

Next we show $\exp_D(v_1) \geq \frac{1}{4}(n^2-1)$. Let D be an asymmetric primitive two-coloured cylce of length n. Since D is primitive, then n must be odd and $n \geq 3$. Furthermore, since D is asymmetric, for v_1 on D there is a $(1,1)^T$-closed walk from v_1 to itself of length 2. Let $p_{1,j}$ be a path which connect v_1 to v_j on D is a $(r_{p_{1,j}}, b_{p_{1,j}})^T$-path consist $r_{p_{1,j}}$ red arcs and $b_{p_{1,j}}$ blue arcs. For v_1 on D, then $(r,b)^T$-walk from v_1 to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} r_{p_{1,j}} \\ b_{p_{1,j}} \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers x_1, x_2 and x_3.
For each v_j on D, there is a $(r,b)^T$-walk from v_1 to v_j. If $n = 4m+1$, we take a $(r,b)^T$-walk from v_1 to $v_{\frac{n-1}{2}}$ and to $v_{\frac{n+1}{2}}$. And if $n = 4m+3$, we take a $(r,b)^T$-walk from v_1 to $v_{\frac{n+1}{2}}$ and to $v_{\frac{n+3}{2}}$. Then we find $\exp_D(v_k) \geq \frac{1}{4}(n^2-1)$.
There are red path of length e and blue path of length f from v_k to $v_{\frac{n-1}{2}}$ if $n = 4m+1$ and to $v_{\frac{n+1}{2}}$ if $n = 4m+3$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} + x_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + x_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers e, f, x_1, x_2 and x_3.
There are also red path of length g and blue path of length h from v_k to $v_{\frac{n+1}{2}}$ if $n = 4m+1$ and to $v_{\frac{n+3}{2}}$ if $n = 4m+3$, then the $(r,b)^T$-walk is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix} + y_1 \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + y_2 \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + y_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for some nonnegative integers g, h, y_1, y_2 and y_3. We note that if $n = 4m+1$, then $e = \frac{n-1}{4}$, $f = 0$, $g = 0$ and $h = \frac{n-1}{4}$. But if $n = 4m+3$, then $e = \frac{n-1}{2}$, $f = \frac{n+1}{4}$, $g = \frac{n+1}{4}$, and $h = \frac{n-1}{2}$. By setting both of those equations we have

$$\begin{bmatrix} e-g \\ f-h \end{bmatrix} = (y_1 - x_1) \begin{bmatrix} \frac{n+1}{2} \\ \frac{n-1}{2} \end{bmatrix} + (y_2 - x_2) \begin{bmatrix} \frac{n-1}{2} \\ \frac{n+1}{2} \end{bmatrix} + (y_3 - x_3) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{2.5}$$

Substraction the second by the first component of Equation 2.5, then we have if $n = 4m+1$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-1}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-1}{4})$ and hence $y_1 \geq \frac{n-1}{4}$ or $x_2 \geq \frac{n-1}{4}$. And if $n = 4m+3$, we have $(y_1 - x_1) + (x_2 - y_2) = 2(\frac{n-3}{4})$. This implies $y_1 + x_2 \geq 2(\frac{n-3}{4})$ and hence $y_1 \geq \frac{n-3}{4}$ or $x_2 \geq \frac{n-3}{4}$. Hence we now have that $\exp_D(v_k) \geq \frac{1}{4}(n^2-1)$. Since we have $\exp_D(v_k) \leq \frac{1}{4}(n^2-1)$ and also $\exp_D(v_k) \geq \frac{1}{4}(n^2-1)$ imply that $\exp_D(v_k) = \frac{1}{4}(n^2-1)$.

For $k \neq 1$, there is a $(r,b)^T$-walk from v_k to v_j that starts from v_k to v_1 and then continue from v_1 to v_j. Let $p_{k,1}$ be a path from v_k to v_1 of length $\lfloor \frac{k}{2} \rfloor$. We have the composition of $(r,b)^T$-walk from v_1 to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} = \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Then we find the $(r,b)^T$-walk from v_k to v_j is of the form

$$\begin{bmatrix} r \\ b \end{bmatrix} \leq \left\lfloor \frac{k}{2} \right\rfloor + \begin{bmatrix} \frac{n^2-1}{8} \\ \frac{n^2-1}{8} \end{bmatrix}.$$

Then we have $\exp_D(v_k) \leq \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.

Theorem 2.3. *Let D be an asymmetric primitive two-coloured cycle of length n, then the exponent v_k of D $exp_D(v_k) = \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.*
Proof. Let D be an asymmetric primitive two-coloured cylce of length n. Since D is primitive, then n must be odd and $n \geq 3$. Lemma 2.1 and Lemma 2.2 imply that $\exp_D(v_k) = \frac{1}{4}(n^2-1) + \lfloor \frac{k}{2} \rfloor$.

REFERENCES

[1] B. L. Shader and S. Suwilo, "Exponent of nonnegative matrix pairs," *Linear Algebra Appl.*, vol. 363, pp. 275–293, 2003.

[2] S. Suwilo, "2-exponents of two-coloured lollipops," *Linear Algebra Appl.*, vol. 21, pp. 11–22, 2008.

[3] Y. Gao and Y. Shao, "Generalized exponents of primitive two-colored digraph," *Linear Algebra Appl.*, vol. 430, pp. 1550–1565, 2009.

[4] V. M. Fomichev and Y. E. Avezova, "The exact formula for the exponents of the mixing digraphs of register transformations," *Journal of Applied and Industrial Mathematics*, vol. 2, 2011.

[5] S. Suwilo, "Vertex exponent of two-colored primitive extermal ministrong digraphs," *Global Journal of Technology and Optimization*, vol. 14, pp. 308–320, 2020.

[6] A. Syahmarani and S. Suwilo, "Vertex exponents of a class of two-colored hamiltonian digraphs," *Journal of the Indonesian Mathematical Society*, vol. 18, pp. 1–19, 2012.

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