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The Cesàro Operator on Some Sequence Spaces in Riesz Space of Non-Absolute Type

***E. Herawati*[1*] *and Supama*[2]**

[1]*Department of Mathematics, Universitas Sumatera Utara, Medan, 20155, Indonesia*
[2]*Department of Mathematics, Universitas Gadjah Mada, Yogyakarta, 55281, Indonesia*

Abstract. The Cesàro operators C are investigated on the class E-valued sequence spaces $c_0(E)$, $c(E)$ and $\ell_\infty(E)$ with E is a Riesz space. Besides, we also carry out that C are order bounded operators.

Keywords: Banach spaces, Bounded operator, Ideal, Regular operator, Riesz spaces.

Abstrak. *Operator Cesàro C diteliti atas kelas ruang barisan bernilai-E* $c_0(E)$*,* $c(E)$ *dan* $\ell_\infty(E)$ *dengan E merupakan ruang Riesz. Selain itu, juga diperoleh bahwa C merupakan operator terbatas terurut.*

Kata Kunci: *Ruang Banach, Operator terbatas, Ideal, Operator regular, Ruang Riesz.*

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1 Introduction

The operator that acting on sequences on functions based on an averaging process is called the Cesàro operator. This operator have been studied in a variety of Banach spaces [1, 2]. For the Fréchet spaces, that is generalization of Banach spaces, this operator also studied [3]. Moreover, by using the convergence topology in non-negative real number, the Cesàro operator studied in the real non-negative sequence spaces and in the quojection Fréchet spaces [4]. When X has an order structure, Herawati [5] introduced space of sequence in X and studied some properties by using the order convergence. In the present paper, by using the certain type of the space in [6]. We analyze the order continuous of the Cesàro operator and show this operators are order bounded.

Throughout the sequel, we denotes E as a Riesz space and its cone positive by E^+. A Riesz space E is called *order complete*, if every non-empty subset of E has a supremum and an infimum. The notation $x_n \downarrow$ is used for decreasing sequence in E and $x_n \downarrow x_0$ proved $x_n \downarrow$ and $\inf(x_n) = x_0$ exists in E.

*Corresponding author at: Department of Mathematics, Universitas Sumatera Utara, Medan, 20155, Indonesia

E-mail address: elvina@usu.ac.id

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A sequence (x_n) in E is said *order convergent to* x_0 *in* E, if there exists a sequence $(p_n) \subset E$ with $p_n \downarrow \vartheta$ and $|x_n - x_0| \leq p_n$ for each n, written as $x_n \xrightarrow{\vartheta} x_0$. A sequence $(x_n) \subset E$ is said to be *order-Cauchy*, if there exists $p_n \downarrow \vartheta$ in E and $|x_m - x_n| \leq p_n$ for all $m \geq n \geq 1$. If every order-Cauchy sequence is order convergent, then E is called *order-Cauchy complete*. For all this notation, we refer to [7]. We shall write $\Omega(E)$ and $\Phi(E)$ for the vector space consisting of all sequences and finitely non-zero sequence from a vector space E, respectively.

Furthermore, the collection of all linear operators from a Riesz space E to another Riesz space F, denoted by $\mathcal{L}(E,F)$, and the subspace of $\mathcal{L}(E,F)$ consisting of all order bounded operators, denoted by $\mathcal{L}^b(E,F)$. The σ-order continuous operators denoted by $\mathcal{L}^{so}(E,F)$ and this space is a subspace of $\mathcal{L}^b(E,F)$.

Concerning the above spaces, we have

Theorem 1. (see [5])

(i) If E is an order complete, then $\Lambda(E)$ and $\Phi(E)$ is also an order complete.

(ii) The Riesz subspace $\Lambda(E)$ of $\Omega(E)$ order complete if and only if it is ideal in $\Omega(E)$.

Theorem 2. (see [5])
If (T_n) is an increasing or decreasing sequence of operators in $\mathcal{L}^b(E,F)$ such that $T_n x \xrightarrow{\vartheta} Tx$ for $x \in E$ and for some $T \in \mathcal{L}(E,F)$, then

$$T \in \mathcal{L}^b(E,F) \quad \text{and} \quad T_n \xrightarrow{\vartheta} T \quad \text{in} \quad \mathcal{L}^b(E,F).$$

2 Cesàro Operator on Special Case of $\Lambda(E)$

Let E be a Riesz space, the non-absolute type E-valued sequence space is special case of E-valued sequence space $\Lambda(E)$, given as follows

$$\begin{aligned} c_0(E) &= \left\{ x = (x_n) \in \Omega(E) \mid x_n \xrightarrow{\vartheta} \vartheta \quad \text{in} \quad E \right\} \\ c(E) &= \left\{ x = (x_n) \in \Omega(E) \mid \text{there exists } x_0 \in E \text{ such that } x_n \xrightarrow{\vartheta} x_0 \right\} \\ \ell_\infty(E) &= \left\{ x = (x_n) \in \Omega(E) \mid \text{there exists } x_0 \in E \text{ such that } \sup_{n \geq 1}\{x_k\} = x_0 \right\} \end{aligned}$$

It is easy to verify that E-valued sequence space $c_0(E)$ and $\ell_\infty(E)$ are ideals in $\Omega(E)$ and $c(E)$ is a Riesz subspace of $\Omega(E)$. Furthermore, if $x = (x_n) \in \Omega(E)$, the operator $C : \Omega(E) \longrightarrow \Omega(E)$ defined by

$$\begin{aligned} C(x) &= \left(x_1, \frac{x_1 + x_2}{2}, \cdots, \frac{x_1 + x_2 + \cdots + x_n}{n}, \cdots \right) \\ &= \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n=1}^{\infty} \end{aligned}$$

is called *Cesàro Operator*.

Theorem 3.
Let E be a Riesz space and $x=(x_n)\in\Omega(E)$, then $x\in c_0(E)$ if and only if $C(|x|)\in c_0(E)$.
Proof.
For proving the necessity. Let $x\in c_0(E)$, then $x_n \xrightarrow{\vartheta} 0$, so there exists a sequence $p_n\downarrow\vartheta$ in E such that $|x_n|\le p_n$ for each n. Since

$$C(|x|)=\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)_{n=1}^{\infty}$$

and for every $n\in E$

$$\frac{1}{n}\sum_{k=1}^{n}|x_k|\le p_n,$$

it follow that

$$\frac{1}{n}\sum_{k=1}^{n}|x_k| \xrightarrow{\vartheta} 0$$

for every n. Therefore $C'(|x|)\in c_0(E)$.

For sufficiency. If $C(|x|)\in c_0(E)$, then

$$\frac{1}{n}\sum_{k=1}^{n}|x_k| \xrightarrow{\vartheta} 0 \quad \text{in } E.$$

It means, there exists a sequence $p_n\downarrow\vartheta$ in E and for every n,

$$\frac{1}{n}\sum_{k=1}^{n}x_k\le\frac{1}{n}\sum_{k=1}^{n}|x_k|\le p_n.$$

So $x=(x_n)\in c_0(E)$. ■

In the similar ways, we could prove the following theorem.

Theorem 4.
If $x=(x_n)\in\Omega(E)$ and $\Lambda=\{c,\ell_\infty\}$, $x\in\Lambda(x)$ if and only if $C(|x|)\in\Lambda(x)$.

3 Order Boundedness

The proof of the Cesàro operators C on $\Lambda(E)$ order bounded, can be made to depend on the Riesz space E is an order complete.

Theorem 5.
Let $\Lambda(E)$ be a Riesz subspace of $\Omega(E)$ and C be a Cesàro operator on $\Lambda(E)$, then:

(i) $C\in\mathcal{L}^{+}\Big(\Lambda(E)\Big)$

(ii) $C\in\mathcal{L}^{b}\Big(\Lambda(E)\Big)$

Proof.
For $\Lambda=c_0$. It is easy to verify (i) to proof point (ii). So, we just prove the point (ii). Since $c_0(E)$ is an ideal of $\Omega(E)$, it follow that it is order-Cauchy complete.

Furthermore, for any $x \in c_0(E)$, the Cesàro operator C on $c_0(E)$ is a sequence in E,

$$C(x) = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right).$$

Since

$$\frac{1}{n} \sum_{k=1}^{n} x_k \leq \sum_{k=1}^{n} |x_k|$$

and for $x \in c_0(E)$

$$\sum_{k=1}^{n} |x_k| \longrightarrow 0,$$

we can define a sequence

$$(z_n) = \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)$$

with $z_n \geq 0$ for every n.

Define the operator $T_1 : c_0(x) \longrightarrow c_0(x)$ by

$$x \longmapsto T_1 x = \left(\sum_{k=1}^{n} |x_k| \right)_n$$

Then $T_1 \in \mathcal{L}^+\Big(c_0(E)\Big)$ and for every $x \in c_0(E)$

$$C(x) \leq T_1(x).$$

It means $C \in \mathcal{L}^r\Big(c_0(E)\Big)$, that is the Cesàro operator C on $c_0(E)$ is a regular operator. Since $c_0(E)$ is an order-Cauchy complete, it follow that $C \in \mathcal{L}^b\Big(\Lambda(E)\Big)$.

For $\Lambda = \ell_\infty$, then $\ell_\infty(E)$ is an ideal of $\Omega(E)$. If $x \in \ell_\infty(E)$ is given, then Cesàro operator

$$C(|x|) = \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right) \in \ell_\infty(E).$$

Since for every $x \in \ell_\infty(E)$

$$\frac{1}{n} \sum_{k=1}^{n} |x_k| \leq \frac{2}{n} \sum_{k=1}^{n} |x_k|,$$

and if we define the operator $T_2 : \ell_\infty(E) \longrightarrow \ell_\infty(E)$ by

$$T_2 = \left(\frac{2}{n} \sum_{k=1}^{n} |x_k| \right)_{n=1}^{\infty},$$

then $T_2 \in \mathcal{L}^+\Big(\ell_\infty(E)\Big)$ and for every $x \in \ell_\infty(E)$

$$C(x) \leq T_2(x).$$

Since $\ell_\infty(E)$ is an order-Cauchy complete, then $C \in \mathcal{L}^b\Big(\ell_\infty(E)\Big)$. ■

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