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On Sequentially Defined Function Spaces and Bounded Linear Functionals

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Abstract. In this paper, we construct a sequentially defined function space $L_{p,q}(\Omega)$ and observe its topological properties. Further, we formulate necessary and sufficient conditions for bounded linear functionals on the space.

Keywords: Hölder's Inequality, Linear Functional, Minkowsky's Inequality, Sequentially Defined.

Abstrak. *Pada paper ini, kami mendefinisikan suatu ruang dari fungsi-fungsi terukur Lebesgue secara barisan $L_{p,q}(\Omega)$ dan mempelajari sifat-sifat topologinya. Lebih lanjut, kami merumuskan kondisi-kondisi perlu dan cukup untuk fungsional linear terbatas pada ruang ini.*

Kata Kunci: *Ketaksamaan Hölder, Fungsional Linear, Ketaksamaan Minkowsky, Terdefinisi secara Barisan.*

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1 Introduction

In modern analysis, mathematicians investigate functions by considering a set of functions. In this case, they consider a function as a point in the set [1]. Generally, investigation on the set is done related to geometric properties and algebraic structures of the set. A set of functions with some certain conditions is called a function space.

Function spaces have very crucial applications and play important roles in many areas, such as optimisation, economics, engineering, physics, and so on. Mathematicians observed that different problems from various fields often have related features and properties of some function spaces. Because of this reason, the topics of function spaces gain a lot of attention of many researchers from many areas.

Very fundamental function spaces are the spaces L_p, $1 \leq p \leq \infty$. They are known as the Lebesgue spaces [2]. Some geometric and topological properties of the spaces L_p have discussed in [3, 4, 5, 6, 7, 8]. In the references, a representation theorem for a linear functional on the Lebesgue space is also presented. The study of function spaces can not be separated from sequence spaces [5, 9]. Sequence spaces which are closely related to the Lebesgue spaces L_p are the sequence spaces ℓ_p, $1 \leq p \leq \infty$.

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Recently, the theory of function spaces and sequence spaces grow rapidly. Some researchers have made their significant contributions in developing the theory of sequence spaces into vector valued sequence spaces [10, 11, 12]. In this paper, we construct the sequentially defined function spaces $L_{p,q}(\Omega)$ as a generalization of the Lebesgue spaces L_p and the sequence spaces ℓ_p, and observe some their topological properties. We also formulate a representation theorem for a linear functional on the spaces.

2 The Space $L_{p,q}(\Omega)$

Let $\Omega \subset \mathbb{R}$ be a Lebesgue measurable set and (Ω, Σ, m) a measure space. Throughout this paper, the symbols $M(\Omega)$ and $\omega(\Omega)$ denote a collection of all measurable functions from Ω into $\mathbb{R}^*$ and a collection of all sequences $\{f_n\}$ in $M(\Omega)$, respectively. For any $f = \{f_n\} \in \omega(\Omega)$ and $1 \leq p < \infty$, we define

$$\begin{aligned} f^p &= \{f_n^p\}, \text{ and} \\ |f| &= \{|f_n|\}. \end{aligned}$$

The conjugate of p is a real number p' such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Let $1 \leq p, q < \infty$. We define a sequentially defined function space

$$L_{p,q}(\Omega) = \left\{ f = \{f_n\} \in \omega(\Omega) : \sum_{n=1}^{\infty} \left(\int_\Omega |f_n|^p \right)^{\frac{q}{p}} < \infty \right\}.$$

It can be proved that $L_{p,q}(\Omega)$ is a linear space over the real number system $\mathbb{R}$. The definition of $L_{p,q}(\Omega)$ gives a straight consequence as stated in the following lemma.

Lemma 1. A sequence $f = \{f_n\} \in L_{p,q}(\Omega)$ if and only if $f_n \in L_p(\Omega)$ for every $n \in \mathbb{N}$ and $\{(\int_\Omega |f_n|^p)^{\frac{1}{p}}\} \in \ell_q$.

Now, let us define a function $\|.\|_{p,q} : L_{p,q}(\Omega) \to \mathbb{R}$ by

$$\|f\|_{p,g} = \left(\sum_{n=1}^{\infty} \left(\int_\Omega |f_n|^p \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}, \quad f \in L_{p,q}(\Omega).$$

Then, we have the following theorems.

Theorem 2. If $f \in L_{p,q}(\Omega)$ and $g \in L_{p',q'}(\Omega)$, then $fg \in L_{1,1}(\Omega)$ and

$$\sum_{n=1}^{\infty} \int_\Omega |f_n g_n| \leq \sum_{n=1}^{\infty} \left(\int_\Omega |f_n|^p \right)^{\frac{1}{p}} \left(\int_\Omega |g_n|^{p'} \right)^{\frac{1}{p'}} \leq \|f\|_{p,q}.\|g\|_{p',q'}$$

Proof. Following the Lemma 1., $f_n \in L_p(\Omega)$ and $g_n \in L_{p'}(\Omega)$ for every $n \in \mathbb{N}$. So, the Hölder's inequality implies $f_n g_n \in L_1(\Omega)$ and

$$\int_\Omega |f_n g_n| \leq \left(\int_\Omega |f_n|^p \right)^{\frac{1}{p}} . \left(\int_\Omega |g_n|^{p'} \right)^{\frac{1}{p'}},$$

for every $n \in \mathbb{N}$. Further, if for any $n \in \mathbb{N}$, we define

$$a_n = \Big(\int_\Omega |f_n|^p \Big)^{\frac{1}{p}},$$

and

$$b_n = \Big(\int_\Omega |g_n|^{p'} \Big)^{\frac{1}{p'}}$$

then Lemma 1. and the Hölder's inequality are followed by $\{a_n b_n\} \in \ell_1$ and

$$\sum_{n=1}^{\infty} \Big(\int_\Omega |f_n|^p \Big)^{\frac{1}{p}} \Big(\int_\Omega |g_n|^{p'} \Big)^{\frac{1}{p'}} \leq \|f\|_{p,q}.\|g\|_{p',q'}$$

These completes the proof. ■

Theorem 3. If $f, g \in L_{p,q}(\Omega)$, then

$$\|f+g\|_{p,q} \leq \|f\|_{p,q} + \|g\|_{p,q}$$

Proof. The assertion follows from Theorem 2. and the Minkowsky's inequality. ■

As a straight consequence of the Theorem 3., then we obtain that $L_{p,q}(\Omega)$ is a normed space with respect to $\|.\|_{p,g}$. Now, we are going to prove that the normed space $(L_{p,q}(\Omega), \|.\|_{p,g})$ is complete.

Theorem 4. The normed space $(L_{p,q}(\Omega), \|.\|_{p,g})$ is complete.

Proof. Let $\{f^{(n)}\}$ be a Cauchy sequence in $L_{p,q}(\Omega)$. For any $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any $m, n \geq n_0$,

$$\left(\sum_{k=1}^{\infty} \Big(\int_\Omega |f_k^{(n)} - f_k^{(m)}|^p \Big)^{\frac{q}{p}} \right)^{\frac{1}{q}} = \|f^{(n)} - f^{(m)}\|_{p,g} < \varepsilon$$

This implies,

$$\|f_k^{(n)} - f_k^{(m)}\|_p < \varepsilon,$$

for $m, n \geq n_0$ and for each $k \in \mathbb{N}$. Hence, $\{f_k^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_p(\Omega)$ for every $k \in \mathbb{N}$. Following the completeness of $L_p(\Omega)$, there exists an $f_k \in L_p(\Omega)$ such that

$$f_k(x) = \lim_{n \to \infty} f_k^{(n)}(x),$$

for almost all $x \in \Omega$. Let $f = \{f_k\}$, then

$$\sum_{k=1}^{\infty} \Big(\int_\Omega |f_k|^p \Big)^{\frac{q}{p}} = \lim_{n \to \infty} \sum_{k=1}^{\infty} \Big(\int_\Omega |f_k^{(n)}|^p \Big)^{\frac{q}{p}} < \infty$$

It means $f \in L_{p,q}(\Omega)$. Furthermore,

$$\begin{aligned}\lim_{n\to\infty}\|f^{(n)}-f\|_{p,q} &= \lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\left(\int_{\Omega}|f_k^{(n)}-f_k|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \\ &= \lim_{m,n\to\infty}\left(\sum_{k=1}^{\infty}\left(\int_{\Omega}|f_k^{(n)}-f_k^{(m)}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} = 0\end{aligned}$$

These completes the proof. ■

Theorem 5. Let $\Omega \subset \mathbb{R}$ be a measurable set with $m(\Omega) < \infty$. Given $f \in L_{p,q}(\Omega)$ and $\varepsilon > 0$, then there exists a sequence of bounded measurable functions $g \in L_{p,q}(\Omega)$ such that $\|f-g\|_{p,q} < \varepsilon$.

Proof. For any $n \in \mathbb{N}$, we define a sequence of functions $f^{(n)} = \{f_k^{(n)}\}$ by

$$f_k^{(n)}(x) = \begin{cases} n2^{-k} & , n2^{-k} < f_k(x) \\ f_k(x) & , -n2^{-k} \le f_k(x) \le n2^{-k} \\ -n2^{-k} & , f_k(x) < -n2^{-k} \end{cases}$$

Then $|f_k^{(n)}| \le n2^{-k}$ and $f^{(n)} \in L_{p,q}(\Omega)$. Further, since $|f_k - f_k^{(n)}| \to 0$ a.e. on Ω, then

$$\|f-f^{(n)}\|_{p,q} = \left(\sum_{k=1}^{\infty}\left(\int_{\Omega}|f_k-f_k^{(n)}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \to 0$$

Thus, for any $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\|f-f^{(N)}\|_{p,q} < \varepsilon$. Hence, by taking $g = f^{(N)}$ the assertion follows. ■

3 Bounded Linear Functionals on the Space $L_{p,q}(\Omega)$

We begin this section by proving the following theorem.

Theorem 6. Let $1 < p,q < \infty$. Then for any $g \in L_{p',q'}(\Omega)$, there exists a bounded linear functional F_g on $L_{p,q}(\Omega)$ such that

$$F_g(f) = \sum_{n=1}^{\infty}\int_{\Omega} f_n g_n$$

Moreover, we have $\|F\| \le \|g\|_{p',q'}$.

Proof. Let $g \in L_{p',q'}(\Omega)$. Theorem 2. inform us that

$$\sum_{n=1}^{\infty}\int_{\Omega}|f_n g_n| < \infty,$$

for every $f = \{f_n\} \in L_{p,q}(\Omega)$. Hence, we can define a functional F_g on $L_{p,q}(\Omega)$ by

$$F_g(f) = \sum_{n=1}^{\infty}\int_{\Omega} f_n g_n, \quad f \in L_{p,q}(\Omega)$$

It is clearly that F_g is linear. By Theorem 2., for any $f \in L_{p,q}(\Omega)$ we have

$$|F_g(f)| \leq \|f\|_{p,q}.\|g\|_{p',q'},$$

therefore F is bounded and $\|F\| \leq \|g\|_{p',q'}$. ■

The main goal of this section is to prove that the converse of the theorem holds. For proving the goal, we need the following theorems.

Theorem 7. Let $1 < p,q < \infty$ and F a bounded linear functional on $L_{p,q}([0,1])$. Then there exists a sequence $g = \{g_n\}$ with $g_n \in L_{p'}([0,1])$ for every $n \in \mathbb{N}$ such that

$$F(f) = \sum_{n=1}^{\infty} \int_0^1 f_n g_n$$

Moreover, we also have $\|F\| \leq \sum_{n=1}^{\infty} \|g_n\|_{p'}$.

Proof. For each $k \in \mathbb{N}$, let F_k be a bounded linear functional on $L_p([0,1])$ such that

$$F_k(s) = F(0,0,\ldots,s,\ldots),$$

where $(0,0,\ldots,s,\ldots)$ is a sequence of p-integrable functions with the k^{th}-element is s and the others are $0s$. By the Riesz's Representation Theorem, there exists a function $g_k \in L_{p'}([0,1])$ such that

$$F_k(s) = \int_0^1 s g_k$$

Let $f \in L_{p,q}([0,1])$ be an arbitrarily, then following Lemma 1., $f = \{f_k\}$ for some $f_k \in L_p([0,1])$, $k \in \mathbb{N}$. Since F is linear, then

$$F(f) = \sum_{k=1}^{\infty} \int_0^1 f_k g_k$$

Now, since $\|f_k\|_p \leq \|f\|_{p,q}$, then

$$|F(f)| \leq \sum_{k=1}^{\infty} \int_0^1 |f_k g_k| \leq \|f\|_{p,q} \sum_{k=1}^{\infty} \|g_k\|_{p'}$$

Hence,

$$\|F\| \leq \sum_{k=1}^{\infty} \|g_k\|_{p'}.$$

These completes the proof. ■

Theorem 8. Let $1 < p,q < \infty$ be such that $\frac{q}{p} = \frac{q'}{p'}$ and g_n be an integrable function on a set of finite measure Ω for every $n \in \mathbb{N}$. If there exists a constant $M > 0$ such that

$$\sum_{n=1}^{\infty} \left(\int_{\Omega} f_n g_n \right)^{\frac{q'}{p'}} \leq M \|f\|_{p,q},$$

for every sequence of bounded function $f \in L_{p,q}(\Omega)$, then $g \in L_{p',q'}(\Omega)$ and $\|g\|_{p',q'} \leq M$.

Proof. For any $n \in \mathbb{N}$, we define sequences $g^{(n)}$ and $f^{(n)}$ as follow

$$g_k^{(n)}(x) = \begin{cases} g_k(x) & , |g_k(x)| \leq n2^{-k} \\ \\ 0 & , |g_k(x)| > n2^{-k} \end{cases}$$

and

$$f_k^{(n)} = |g_k^{(n)}|^{\frac{p'}{p}} .sgn\left(g_k^{(n)}\right).$$

Then $|f_k^{(n)}|^p = |g_k^{(n)}|^{p'}$ and $f_k^{(n)} g_k = f_k^{(n)} g_k^{(n)} = |g_k^{(n)}|^{p'}$. So,

$$\sum_{k=1}^{\infty} \left(\int_\Omega |f_k^{(n)}|^p\right)^{\frac{q}{p}} = \sum_{k=1}^{\infty} \left(\int_\Omega |g_k^{(n)}|^{p'}\right)^{\frac{q}{p}} < \infty,$$

i.e. $f^{(n)} \in L_{p,q}(\Omega)$. We have also,

$$\|f^{(n)}\|_{p,q} = \left(\sum_{k=1}^{\infty} \left(\int_\Omega |g_k^{(n)}|^{p'}\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q}}$$

Further, by the hypothesis we obtain

$$\sum_{k=1}^{\infty} \left(\int_\Omega |g_k^{(n)}|^{p'}\right)^{\frac{q'}{p'}} = \sum_{k=1}^{\infty} \left(\int_\Omega f_k^{(n)} g_k\right)^{\frac{q'}{p'}} \leq M\|f^{(n)}\|_{p,q} = M\left(\sum_{k=1}^{\infty} \left(\int_\Omega |g_k^{(n)}|^{p'}\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q}}$$

Hence,

$$\left(\sum_{k=1}^{\infty} \left(\int_\Omega |g_k^{(n)}|^{p'}\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}} \leq M$$

Since, $g_k^{(n)}$ converges to g_k for every $k \in \mathbb{N}$, then by the Fatou's Lemma we obtain $g \in L_{p',q'}(\Omega)$ and $\|g\|_{p',q'} \leq M$. ∎

We are now in a position to prove the following theorem that gives a characterization of any bounded linear functional on $L_{p,q}([0,1])$.

Theorem 9. Let $1 < p, q < \infty$ and F a bounded linear functional on $L_{p,q}([0,1])$. Then there exists a sequence $g \in L_{p',q'}([0,1])$ such that

$$F(f) = \sum_{n=1}^{\infty} \int_0^1 f_n g_n$$

Moreover, we also have $\|F\| \leq \|g\|_{p',q'}$.

Proof. Let χ_s be a characteristic function of the interval $[0,s]$ for every $s \in [0,1]$ and $k \in \mathbb{N}$. Define the sequence

$$h_s^{(k)} = \{0, \ldots, 0, \chi_s, 0, 0, \ldots\},$$

where χ_s is the k^{th}-element, then $h_s^{(k)} \in L_{p,q}([0,1])$. For any $k \in \mathbb{N}$, we define a function $\phi_k : [0,1] \to \mathbb{R}$ by

$$\phi_k(s) = F\left(h_s^{(k)}\right)$$

Let $\{(s_i, s_i')\}$ be any finite collection of non overlapping intervals in $[0,1]$. If for any $k \in \mathbb{N}$,

$$f_k = \sum_i \left(h_{s_i}^{(k)} - h_{s_i'}^{(k)}\right) sgn\left(\phi_k(s_i) - \phi_k(s_i')\right),$$

then

$$\sum_i |\phi_k(s_i) - \phi_k(s_i')| = F(f_k)$$

Since, F is bounded, then ϕ_k is absolutely continuous on $[0,1]$ for every $k \in \mathbb{N}$. Hence, there is an integrable function g_k on $[0,1]$ such that

$$\phi_k(s) = \int_0^s g_k,$$

for every $s \in [0,1]$. Thus,

$$F\left(h_s^{(k)}\right) = \int_0^s g_k = \int_0^1 g_k \chi_s$$

Let $f = \{f_n\}$ be a sequence of bounded measurable functions on $[0,1]$ such that $f \in L_{p,q}([0,1])$, then by one of the Littlewood's three principles, for any $n \in \mathbb{N}$, there exists a bounded sequence $\{\psi_k^{(n)}\}$ of step functions that converges almost everywhere to f_n. Since $\{|f_n - \psi_k^{(n)}|\}$ is uniformly bounded and tends to 0 almost everywhere, then the Bounded Convergence Theorem gives us $\|f - \psi_k\|_{p,q} \to 0$, as $k \to \infty$. By linearity and boundedness of F, then we obtain

$$|F(f) - F(\psi_k)| = |F(f - \psi_k)| \leq \|F\|.\|f - \psi_k\|_{p,q}$$

Hence, we must have

$$F(f) = \lim F(\psi_k) = \lim \sum_{n=1}^{\infty} \int_0^1 g_n \psi_k^{(n)}$$

Since $g_n \psi_k^{(n)} \leq |g_n|$ and $\lim_k g_n \psi_k^{(n)} = g_n f_n$ for almost all $x \in [0,1]$, then the Lebesgue Convergence Theorem gives us

$$F(f) = \sum_{n=1}^{\infty} \int_0^1 g_n f_n$$

Now, take any $f \in L_{p,q}([0,1])$, then for any $\varepsilon > 0$ there exists a sequence $\Psi = \{\psi_n\}$ of step functions such that

$$\|f - \Psi\|_{p,q} < \varepsilon$$

Since ψ_n is bounded for each $n \in \mathbb{N}$, then

$$F(\Psi) = \sum_{n=1}^{\infty} \int_0^1 \psi_n g_n$$

Hence,

$$\begin{aligned}\left|F(f)-\sum_{n=1}^{\infty}\int_0^1 f_n g_n\right| &= \left|F(f)-F(\Psi)+\sum_{n=1}^{\infty}\int_0^1 \psi_n g_n-\sum_{n=1}^{\infty}\int_0^1 f_n g_n\right| \\ &\leq |F(f-\Psi)|+\left|\sum_{n=1}^{\infty}\int_0^1 (\psi_n-f_n)g_n\right| \\ &\leq \|F\|.\|f-\Psi\|_{p,q}+\|\Psi-f\|_{p,q}.\|g\|_{p',q'} \\ &< \left(\|F\|+\|g\|_{p',q'}\right)\varepsilon\end{aligned}$$

Since ε is an arbitrary, then

$$F(f)=\sum_{n=1}^{\infty}\int_0^1 f_n g_n$$

We have also $\|F\| \leq \|g\|_{p',q'}$. ■

4 Conclusions

For any Lebesgue measurable set $\Omega \subset \mathbb{R}$ and real numbers $1 \leq p,q < \infty$, we can construct the sequentially defined function spaces $L_{p,q}(\Omega)$. The spaces are Banach spaces with respect to the norm

$$\|f\|_{p,g}=\left(\sum_{n=1}^{\infty}\left(\int_{\Omega}|f_n|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}, \quad f \in L_{p,q}(\Omega)$$

Further, we can formulate a representation theorem for bounded linear functionals on the spaces, as well.

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