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Some Vector Valued Sequence Spaces Generated by Musielak-Phy Function Over 2-Normed Spaces

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Abstract. In this work, we introduced some new vector valued sequence spaces over 2-normed spaces using Musielak-Phy function $\Phi = (\varphi_n)$. We also studied some properties of these spaces.

Keyword: Musielak-Phy Function, Vector Valued Sequence Space, 2-Normed Spaces

Abstrak. *Pada penelitian ini, kami memperkenalkan beberapa ruang barisan bernilai vektor baru atas ruang bernorma-2 menggunakan fungsi Musielak-Phy $\Phi = (\varphi_n)$. Kami juga mempelajari beberapa sifat dari ruang ini.*

Kata Kunci: *Fungsi Musielak-Phy, Ruang Barisan Bernilai Vektor, Ruang Bernorma-2*

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1. Introduction

A phy-function, φ, is a non-negative real valued function on $\mathbb{R}$, which is, continuous, even, non-decreasing function and vanishing at zero. A φ-function is a generalization of Orlicz function. Using the idea of Orlicz function, M, Lindenstrauss and Tzafriri [3] defined the scalar sequence space such that

$$\sum_{k\geq 1} M\left(\frac{|x_k|}{\rho}\right) < \infty$$

for some $\rho > 0$. This space, denoted by ℓ_M, becomes a Banach space which is called an Orlicz sequence space under the following norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k\geq 1} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}.$$

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Let E_k and Y be Banach spaces. The collection of all bounded linear operators from E_k to Y denoted by $B(E_k, Y)$ become a Banach space respected to the following norm

$$\|A_k\| = \sup\{\|A_k z\| : z \in U(E_k)\},$$

with $A_k \in B(E_k, Y)$ and $U(E_k)$ is the closed unit sphere in E_k. By E_k', denotes the collection of all continuous dual of E_k. Srivastava and Ghosh [6] introduced a class of vector valued sequences using Orlicz-function M, i. e. $\ell_M\big(B(E_k, Y)\big)$ and $\ell_M(E_k')$. They studied Kothe-Toeplitz dual, continuous dual, operator representation and weak convergence for these spaces.

A phy-function, φ, is said to satisfy convex property, if for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$ and every $x, y \in X$ implies

$$\varphi(\alpha x + \beta y) \leq \alpha\varphi(x) + \beta\varphi(y).$$

The concept of 2-normed spaces was introduced by Gahler [1] in the mid 1960s and many others such as Gunawan and Mashadi [2] have studied and obtained various results.

Let X be a linear space over the field K. The function $\|\cdot, \cdot\| : X \times X \to \mathbb{R}$ is to be a 2-norm on X if it is satisfying the following properties

(1) $\|x_1, x_2\| = 0$ if and only if x_1 and x_2 are linearly dependent.

(2) $\|x_1, x_2\| = \|x_2, x_1\|$

(3) $\|\alpha x_1, x_2\| = |\alpha|\|x_1, x_2\|$, $\alpha \in \mathbb{R}$

(4) $\|x_1, x_2 + x_3\| \leq \|x_1, x_2\| + \|x_1, x_3\|$ for all $x_1, x_2, x_3 \in X$

and the pair $(X, \|\cdot, \cdot\|)$, written as $X_{\|\cdot,\cdot\|}$, is called a 2-normed space. For example, we may take $X = \mathbb{R}^2$ equipped with the 2-norm defined as

$$\|x_1, x_2\|_E = \left|\det\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right|.$$

This is the same meaning with the area of the parallelogram spanned by the vectors x_1 and x_2. Then, $X_{\|\cdot,\cdot\|}$ is a 2-normed space.

The sequence (x_k) in a 2-normed space $X_{\|\cdot,\cdot\|}$ is said to be converges to L if

$$\lim_{k\to\infty} \|x_k - L, y\| = 0$$

holds if for every $y \in X_{\|\cdot,\cdot\|}$. Furthermore, the sequence (x_k) in the arbitrary 2-normed space $X_{\|\cdot,\cdot\|}$ is called Cauchy sequence if

$$\lim_{k,p\to\infty} \|x_k - x_p, y\| = 0$$

holds for every $y \in X_{\|\cdot,\cdot\|}$. Furthermore, if every Cauchy sequence in the space $X_{\|\cdot,\cdot\|}$ converges to some $L \in X_{\|\cdot,\cdot\|}$, then $X_{\|\cdot,\cdot\|}$ is said to be complete respected to the 2-norm. Any complete n-normed space is said to be 2-Banach space.

Let $\Phi = (\varphi_k)$ be a Musielak-Phy function and let $X_{\|\cdot,\cdot\|}$ be a 2-normed space. Let $\Omega(X_{\|\cdot,\cdot\|})$ be the space of all $X_{\|\cdot,\cdot\|}$-valued sequences $x = (x_k)$ where $x_k \in X_{\|\cdot,\cdot\|}$. Any sublinear space in $\Omega(X_{\|\cdot,\cdot\|})$ is called $X_{\|\cdot,\cdot\|}$-valued sequence space. In the present paper, we define the following spaces for every $y \in X_{\|\cdot,\cdot\|}$:

$$\ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi) = \left\{ x = (x_k) \in \Omega(X_{\|\cdot,\cdot\|}) : (\exists \rho > 0) \sum_{k\geq 1} \varphi_k \left(\left\| \frac{x_k}{\rho}, y \right\| \right) < \infty \right\} \quad (1),$$

$$\ell_\infty^{\exists}(X_{\|\cdot,\cdot\|}, \Phi) = \left\{ x = (x_k) \in \Omega(X_{\|\cdot,\cdot\|}) : (\exists \rho > 0) \sup_k \varphi_k \left(\left\| \frac{x_k}{\rho}, y \right\| \right) < \infty \right\} \quad (2),$$

$$c_0^{\exists}(X_{\|\cdot,\cdot\|}, \Phi) = \left\{ x = (x_k) \in \Omega(X_{\|\cdot,\cdot\|}) : (\exists \rho > 0) \lim_{k\to\infty} \varphi_k \left(\left\| \frac{x_k}{\rho}, y \right\| \right) = 0 \right\} \quad (3).$$

Throughout this paper, we introduce and study vector valued sequence spaces generated by a Musielak-Phy function over 2-normed spaces.

2. Results and Discussion

Theorem 1. Let $\Phi = (\varphi_k)$ be a Musielak-Phy function that satisfy convex property, then the space $\ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$, $\ell_\infty^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$ and $c_0^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$ are linear spaces over the field of complex numbers $\mathbb{C}$.

Proof. Let $x = (x_k) \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$ and $\alpha \in \mathbb{C}$. We will show that $\alpha x \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$. It is clearly for $\alpha = 0$. Assume that $\alpha \neq 0$. Since $x = (x_k) \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$, then there exists $\rho > 0$ such that

$$\sum_{k\geq 1} \varphi_k \left(\left\| \frac{x_k}{\rho}, y \right\| \right) < \infty.$$

Define $\gamma = 2\rho|\alpha|$, then $\frac{|\alpha|}{\gamma} = \frac{1}{2\rho}$. Thus

$$\sum_{k\geq 1} \varphi_k \left(\left\| \frac{\alpha x_k}{\gamma}, y \right\| \right) = \sum_{k\geq 1} \varphi_k \left(\frac{|\alpha|}{\gamma} \|x_k, y\| \right) = \sum_{k\geq 1} \varphi_k \left(\frac{1}{2\rho} \|x_k, y\| \right)$$

$$\leq \frac{1}{2} \sum_{k\geq 1} \varphi_k \left(\left\| \frac{x_k}{\rho}, y \right\| \right) < \infty.$$

Since $\gamma = 2\rho|\alpha| > 0$, then $\alpha x \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$.

Let $\alpha, \beta \in \mathbb{C}$ and $x = (x_k), z = (z_k)$ in $\ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$. We will show that

$\alpha x + \beta y \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$. It is clear if $\alpha = \beta = 0$. Assume that $\alpha \neq 0$ or $\beta \neq 0$. Since $x = (x_k), z = (z_k) \in \ell_1^{\exists}(X_{\|\cdot,\cdot\|}, \Phi)$, then there exists $\rho_1, \rho_2 > 0$ such that

$$\sum_{k\geq 1} \varphi_k \left(\left\| \frac{x_k}{\rho_1}, y \right\| \right) < \infty$$

and

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{z_k}{\rho_2},y\right\|\right)<\infty.$$

We choose $\rho=\sup\{\rho_1,\rho_2\}$. Then

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{\alpha x_k+\beta z_k}{\rho},y\right\|\right)\leq\sum_{k\geq 1}\varphi_k\left(\frac{|\alpha|}{|\alpha|+|\beta|}\left\|\frac{x_k}{\rho},y\right\|+\frac{|\beta|}{|\alpha|+|\beta|}\left\|\frac{z_k}{\rho},y\right\|\right)$$

$$\leq\frac{|\alpha|}{|\alpha|+|\beta|}\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{\rho},y\right\|\right)+\frac{|\beta|}{|\alpha|+|\beta|}\sum_{k\geq 1}\varphi_k\left(\left\|\frac{z_k}{\rho},y\right\|\right)<\infty.$$

It means $\alpha x+\beta y\in\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$. Hence, $\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ is a linear space. With the similar way, we can prove that $\ell_\infty^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ and $c_0^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ is a linear space. ■

Theorem 2. Let $\Phi=(\varphi_k)$ be Musielak-Phy function that satisfy convex property. If $x=(x_k)\in\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ and $y\in X$, then $\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ become a topological linear spaces that normed defined by

$$\|x\|=\inf\left\{\rho>0:\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{\rho},y\right\|\right)\leq 1\right\}.$$

Proof. Firstly, we will show that $\|x\|=0$ if and only if $x=0$. Let $x=0$. Then $x_k=0$ for every natural numbers k. Thus, for every $y\in X_{\|\cdot,\cdot\|}$ and for every $\varepsilon>0$, we get

$$\left\|\frac{x_k}{\varepsilon},y\right\|=\|0,y\|=0.$$

Since Musielak-phy function, Φ, is vanishing at zero, we have for every $k\in\mathbb{N}$,

$$\varphi_k\left(\left\|\frac{x_k}{\varepsilon},y\right\|\right)=\varphi_k(0)=0.$$

Therefore

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{\varepsilon},y\right\|\right)<1.$$

It means $\|x\|<\varepsilon$ for every $\varepsilon>0$. Thus $\|x\|=0$.

Let $\|x\|=0$ for every $x\in\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$. We will show that $x=0$.

Suppose $x_k\neq 0$ for every $k\in\mathbb{N}$. Then $\|x_k,y\|\neq 0$ for every $k\in\mathbb{N}$ and every $y\in X_{\|\cdot,\cdot\|}$. Since $1/n\longrightarrow 0$ as $n\longrightarrow\infty$, then $\|nx_k,y\|=n\|x_k,y\|\longrightarrow\infty$. Since Φ is Musielak-Phy function, then for every $k\in\mathbb{N}$,

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{1/n},y\right\|\right)\to\infty.$$

This is contrary to the fact that $\|x\|=0$. It should be $x_k=0$ for every $k\in\mathbb{N}$ or $x=0$.

Secondly, we will show that $\|\alpha x\|=|\alpha|\|x\|$ for every complex numbers α and $x=(x_k)\in\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$. Since

$$\|\alpha x\|=\inf\left\{\rho>0:\sum_{k\geq 1}\varphi_k\left(\left\|\frac{\alpha x_k}{\rho},y\right\|\right)\leq 1\right\}=\inf\left\{\rho>0:\sum_{k\geq 1}\varphi_k\left(|\alpha|\left\|\frac{x_k}{\rho},y\right\|\right)\leq 1\right\}$$

then, this is clear for $\alpha=0$. Assume that $\alpha\neq 0$. If $\|x\|<\varepsilon$ for every $\varepsilon>0$, then

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{\varepsilon},y\right\|\right)=\sum_{k\geq 1}\varphi_k\left(\left\|\frac{\alpha x_k}{\varepsilon|\alpha|},y\right\|\right)\leq 1.$$

Thus, $\|\alpha x\|\leq|\alpha|\varepsilon$. Therefore $\|\alpha x\|\leq|\alpha|\|x\|$.

Since

$$\|x\|=\left\|\frac{\alpha x}{|\alpha|}\right\|\leq\frac{1}{|\alpha|}\|\alpha x\|$$

for every $\alpha\neq 0$, implies $|\alpha|\|x\|\leq\|\alpha x\|$. We get, $\|\alpha x\|=|\alpha|\|x\|$.

Finally, take any vector $x,z\in\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ and $\alpha,\beta\in(0,1]$ such that $\alpha+\beta=1$, $\|x\|<\alpha$ and $\|z\|<\beta$. Thus, for every $k\in\mathbb{N}$, we get

$$\varphi_k\left(\left\|\frac{x_k+z_k}{\alpha+\beta},y\right\|\right)=\varphi_k\left(\left\|\frac{\alpha}{\alpha+\beta}\frac{x_k}{\alpha}+\frac{\beta}{\alpha+\beta}\frac{z_k}{\beta},y\right\|\right).$$

Since φ_k is a phy-function and it have a convex property implies

$$\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k+z_k}{\alpha+\beta},y\right\|\right)\leq\frac{\alpha}{\alpha+\beta}\sum_{k\geq 1}\varphi_k\left(\left\|\frac{x_k}{\alpha},y\right\|\right)+\frac{\beta}{\alpha+\beta}\sum_{k\geq 1}\varphi_k\left(\left\|\frac{z_k}{\beta},y\right\|\right)$$

$$\leq\frac{\alpha}{\alpha+\beta}+\frac{\beta}{\alpha+\beta}=1.$$

Consequently $\|x+z\|\leq\alpha+\beta$. Thus $\|x+z\|\leq\|x\|+\|z\|$. ■

3. Conclusion

Based on the result section, $\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$, $\ell_\infty^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ and $c_0^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ are vector valued sequence spaces over 2-normed space with Musielak-phy function $\Phi=(\varphi_k)$ satisfying convex property. Furthermore, for specified norm, $\ell_1^{\exists}(X_{\|\cdot,\cdot\|},\Phi)$ be a topological linear spaces.

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